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Multiobjective gradient descent method for fuzzy optimization

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Abstract. Fuzzy multiobjective optimization problems naturally model decision-making situations involving imprecision and uncertainty. This study develops a gradient descent method for finding Pareto-optimal solutions to unconstrained fuzzy multiobjective optimization problems with continuously gH-differentiable objective functions. We establish theoretical connections between weakly Pareto-optimal solutions and Pareto critical points, thereby providing necessary optimality conditions. The proposed steepest descent algorithm computes descent directions by solving a quadratic subproblem and determines step sizes via Armijo line search. Convergence analysis demonstrates that all accumulation points of the iterative sequence are Pareto critical points under convexity assumptions. The method exhibits polynomial computational complexity, ensuring practical efficiency. Numerical experiments on benchmark problems validate the approach, showing superior performance compared to the Newton method for fuzzy multiobjective optimization. The α-level parameterization provides greater control over solution quality, allowing fine-tuning of the trade-off among objectives. Results demonstrate successful convergence for problems with up to three objectives and complex nonlinear fuzzy functions, confirming the method's robustness and effectiveness for real-world applications.

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1. INTRODUCTION

Multiobjective optimization problems arise naturally in many practical applications where multiple, often conflicting, objectives must be optimized simultaneously. Rather than seeking a single optimal solution, multiobjective optimization aims to identify a set of Pareto-optimal solutions, trade-off solutions in which no objective can be improved without degrading at least one other objective. This framework has been extensively studied and has given rise to numerous solution methodologies over the past decades [2, 19, 10].

In real-world applications, uncertainty and imprecision are inherent characteristics of the data due to measurement limitations, incomplete information, or subjective judgments. Fuzzy set theory provides a natural mathematical framework for representing and handling such imprecision [18, 9]. This leads to fuzzy multiobjective optimization problems, where objective functions take fuzzy values, requiring specialized solution approaches that can accommodate both the multiobjective nature and the fuzzy uncertainty.

Several numerical methods have been developed to solve fuzzy multiobjective optimization problems. Wu[22, 23] introduced scalarization techniques and established Karush-Kuhn-Tucker optimality conditions for fuzzy multiobjective problems. More recently, Ghaznavi et al. [4] developed a Newton method to capture weakly Pareto-optimal solutions of unconstrained fuzzy multiobjective optimization problems. However, gradient descent methods -despite their fundamental importance in classical optimization and their proven effectiveness in single-objective fuzzy optimization [14, 8, 7] - have not been systematically explored for fuzzy multiobjective problems. Mondal and Ghosh [11] developed a steepest descent method for interval-valued multiobjective optimization, but extending it to fuzzy-valued objectives introduces additional theoretical and computational challenges.

This research addresses this gap by developing a gradient-based method for unconstrained fuzzy multiobjective optimization problems. Our approach leverages generalized Hukuhara (gH) differentiability to define Pareto critical points and establish connections with weakly Pareto-optimal solutions. The main contributions of this paper are threefold: (i) we establish theoretical relationships between weakly Pareto optimal solutions and Pareto critical points under gH-differentiability assumptions; (ii) we propose a steepest descent algorithm with proven global convergence properties and polynomial computational complexity; and (iii) we demonstrate through numerical experiments that the method achieves superior performance compared to existing approaches.

The remainder of this paper is organized as follows. Section 2 introduces preliminary concepts from fuzzy set theory. Section 3 defines the fuzzy multiobjective optimization problem and solution concepts. Section 4 develops the gradient descent method, including the computation of descent directions and step sizes. Section 5 presents convergence analysis and complexity results. Section 6 provides numerical experiments on benchmark problems. Section 7 discusses the results and comparisons with existing methods. Finally, Section 8 concludes the paper and outlines future research directions.

2. Preliminaries

In this section, we recall some fundamental notions of fuzzy set theory that will be used throughout the paper.
In what follows, unless otherwise specified, $\mathcal{X}$ denotes a subset of $\mathbb{R}^m$ and $\mathfrak{F}$ represents a fuzzy set.

Definition 2.1. ([18, 9]) Let $\mathcal{X}$ be a universe and let $x \in \mathcal{X}$. A fuzzy set $\tilde{A}$ defined on $\mathcal{X}$ is the set of ordered pairs given by

$$\tilde{A} = \{ (x, \mu_{\tilde{A}}(x)) \mid x \in \mathcal{X} \},$$

where $\mu_{\tilde{A}}$ represents the membership degree of x in $\tilde{A}$.

Definition 2.2. [13] Let $\tilde{A}$ be a fuzzy set defined on a universe $\mathcal{X}$ and let $\alpha \in [0, 1]$. The α-cut of $\tilde{A}$, denoted $[\tilde{A}]^\alpha$, is the crisp set defined by:

$$\tilde{A}_\alpha = \{x \in \mathcal{X} \mid \mu_{\tilde{A}}(x) \geq \alpha\} \tag{2.1}$$

Definition 2.3. ([21, 20]) Let $\tilde{A}$ be a fuzzy subset of $\mathbb{R}$. Then $\tilde{A}$ is called a fuzzy number if it is normal, convex, and its α-level set, denoted by $\tilde{A}_\alpha$, is bounded for all $\alpha \in [0, 1]$.

Definition 2.4. ([12]) Let $\tilde{\mathcal{G}} : \mathbb{R}^m \to \mathfrak{F}$ be a fuzzy function. If $\tilde{\mathcal{G}}$ is a triangular fuzzy function defined by $\tilde{\mathcal{G}}(x) = (g_1(x), g_2(x), g_3(x))$, then its α-cut is given by $\tilde{\mathcal{G}}^\alpha(x) = [\mathcal{G}_L^\alpha(x), \mathcal{G}_R^\alpha(x)]$, where $\mathcal{G}_L^\alpha(x) = \alpha(g_2(x) - g_1(x)) + g_1(x)$ and $\mathcal{G}_R^\alpha(x) = \alpha(g_2(x) - g_3(x)) + g_3(x)$.

Definition 2.5. ([3, 17]) Let $\tilde{\mathtt{u}}$ and $\tilde{\mathtt{v}}$ be two fuzzy numbers. The generalized Hukuhara difference between these two numbers is the fuzzy number $\tilde{\mathtt{s}}$, if it exists, given by

$$\tilde{\mathtt{u}} \ominus_{gH} \tilde{\mathtt{v}} = \tilde{\mathtt{s}} \Longrightarrow \begin{cases} (i) & \tilde{\mathtt{u}} = \tilde{\mathtt{v}} \oplus \tilde{\mathtt{s}}, \\ or & \\ (ii) & \tilde{\mathtt{v}} = \tilde{\mathtt{u}} \oplus (-1) \odot \tilde{\mathtt{s}}. \end{cases}$$

In terms of α-cuts, this relation is expressed as follows:

$$\tilde{\mathtt{u}} \ominus_{gH} \tilde{\mathtt{v}} = [\min\{\mathtt{u}_L^\alpha - \mathtt{v}_L^\alpha, \mathtt{u}_R^\alpha - \mathtt{v}_R^\alpha\}, \max\{\mathtt{u}_L^\alpha - \mathtt{v}_L^\alpha, \mathtt{u}_R^\alpha - \mathtt{v}_R^\alpha\}],$$

where $\oplus$ and $\ominus_{gH}$ represent, respectively, the sum and the generalized Hukuhara difference between two fuzzy quantities.

Definition 2.6. ([6]) Let $\tilde{\mathcal{G}} : \mathcal{X} \longrightarrow \mathfrak{F}$. The gH-directional derivative of $\tilde{\mathcal{G}}$ at κ in the direction of the vector $h \in \mathbb{R}^m$, if it exists, is given by

$$\tilde{\mathcal{G}}'_{gH}(\kappa) = \lim_{\mu \to 0^+} \frac{\tilde{\mathcal{G}}(\kappa + \mu h) \ominus_{gH} \tilde{\mathcal{G}}(\kappa)}{\mu}, \quad h \in \mathbb{R}^m.$$

Definition 2.7. ([5]) Let $\tilde{\mathcal{G}} : \mathcal{X} \subseteq \mathbb{R}^m \to \mathfrak{F}$ be a fuzzy function. The gH-gradient of $\tilde{\mathcal{G}}$ at κ is given by

$$\nabla_{gH}\tilde{\mathcal{G}}(\kappa) = \left(\frac{\partial_{gH}\tilde{\mathcal{G}}}{\partial x_1}(\kappa), \frac{\partial_{gH}\tilde{\mathcal{G}}}{\partial x_2}(\kappa), \cdots, \frac{\partial_{gH}\tilde{\mathcal{G}}}{\partial x_m}(\kappa) \right),$$

where $\dfrac{\partial_{gH}\tilde{\mathcal{G}}}{\partial x_j}(\kappa)$, $j \in \{1, 2, \ldots, m\}$, represents the j-th gH-derivative of $\tilde{\mathcal{G}}$ at κ. In this work, this derivative will be denoted by $D^j_{gH}\,\tilde{\mathcal{G}}$.

Definition 2.8. ([15]) A norm on $\mathfrak{F}$ is a mapping $\|\cdot\|_{\mathfrak{F}} : \mathfrak{F} \longrightarrow \mathbb{R}_+$, defined for a fuzzy number $\tilde{\mathtt{u}}$ by

$$\|\tilde{\mathtt{u}}\|_{\mathfrak{F}} = \sup_{\alpha\in[0,1]} \max\{|\mathtt{u}^{\alpha}_L|,\, |\mathtt{u}^{\alpha}_R|\}. \tag{2.2}$$

Lemma 2.9. [16] *Let $\mathcal{X} \subseteq \mathbb{R}^m$ be an open set. If $\tilde{\mathcal{H}} : \mathcal{X} \to \mathfrak{F}$ is a fuzzy function that is gH-differentiable at the point $x^* \in \mathcal{X}$, then there exist positive constants λ and γ such that*

$$\lim_{\lambda\to 0} \frac{1}{\lambda} \odot \left(\tilde{\mathcal{H}}^{\alpha}(x^* - \lambda\vartheta) \ominus_{gH} \tilde{\mathcal{H}}^{\alpha}(x^*)\right) = L^{\alpha}_{x^*}(\vartheta), \forall\, \vartheta \in \mathbb{R}^m \text{ and } |\lambda|\|\vartheta\| < \gamma,$$

where $L^{\alpha}_{x^*}(\vartheta)$ is the α-cut of a linear fuzzy function.
Moreover, if the fuzzy gH-gradient of $\tilde{\mathcal{H}}$ exists at a point $x^* \in \mathcal{X}$, then the linear fuzzy function L_{x^*} is given by

$$L^{\alpha}_{x^*}(\vartheta) = \left(\nabla_{gH}\tilde{\mathcal{H}}^{\alpha}(x^*)\right)^T \odot \vartheta, \quad \forall\, \vartheta = (\vartheta_1, \vartheta_2, ..., \vartheta_m) \in \mathbb{R}^m,$$

where $\left(\nabla_{gH}\tilde{\mathcal{H}}^{\alpha}(x^*)\right)^T \odot \vartheta = \bigoplus_{j=1}^{m} D^j_{gH}\tilde{\mathcal{H}}^{\alpha}(x^*) \odot \vartheta_j$.

Lemma 2.10. [16] *Let $\mathcal{X} \subseteq \mathbb{R}^m$ be a convex set. If $\tilde{\mathcal{H}} : \mathcal{X} \to \mathfrak{F}$ is a convex fuzzy function that is gH-differentiable on $\mathcal{X}$, then we have*

$$\tilde{\mathcal{H}}^{\alpha}(y) \succeq \tilde{\mathcal{H}}^{\alpha}(x) \oplus \left(\nabla_{gH}\tilde{\mathcal{H}}^{\alpha}(x)\right)^T \odot (y - x), \quad \forall\, x,\ y \in \mathcal{X}.$$

Definition 2.11. ([1]) Let $\mathcal{X}$ be a convex subset of $\mathbb{R}^m$. A fuzzy function $\tilde{\mathcal{H}} : \mathcal{X} \to \mathfrak{F}$ is said to be convex if for all vectors x_1 and x_2 in $\mathcal{X}$, we have

$$\tilde{\mathcal{H}}(\mu_1 x_1 + \mu_2 x_2) \preceq \mu_1 \odot \tilde{\mathcal{H}}(x_1) \oplus \mu_2 \odot \tilde{\mathcal{H}}(x_2), \quad \forall\, \mu_1, \mu_2 \in [0, 1] \text{ such that } \mu_1 + \mu_2 = 1.$$

Property 2.12. ([1]) $\tilde{\mathcal{H}}$ is convex if the functions $\mathcal{H}^{\alpha}_L$ and $\mathcal{H}^{\alpha}_R$ are convex for all $\alpha \in [0, 1]$.

3. Fuzzy multiobjective optimization problem

An unconstrained fuzzy multiobjective optimization problem is formulated as follows

$$\min_{x} \tilde{\mathcal{H}}(x), \tag{3.1}$$

where $\tilde{\mathcal{H}} : \mathbb{R}^m \longrightarrow \mathfrak{F}^n$ is a fuzzy multiobjective function whose components $\tilde{\mathcal{H}}_i : \mathbb{R}^m \longrightarrow \mathfrak{F}$ are gH-continuously differentiable for all $i \in \{1, 2, \ldots, n\}$.

Remark 3.1. To solve such a problem, we defuzzify the multiobjective function, which amounts to associating its α-cut with it. Thus, to the fuzzy objective function, we associate the function $\tilde{\mathcal{H}}^{\alpha}$, which is an interval-valued function.

We now turn our attention to the notion of solutions for problem (3.1).

Definition 3.2. A point $\mathring{x}$ is weakly Pareto optimal for problem (3.1) if there does not exist another point $x \in \mathbb{R}^m$ such that $\tilde{\mathcal{H}}_i^{\alpha}(x) \prec \tilde{\mathcal{H}}_i^{\alpha}(\mathring{x})$ for all $i \in \{1, 2, \ldots, n\}$.

Definition 3.3. A point $\mathring{x}$ is Pareto optimal for problem (3.1) if there does not exist another point $x \in \mathbb{R}^m$ such that $\tilde{\mathcal{H}}_i^{\alpha}(x) \preceq \tilde{\mathcal{H}}_i^{\alpha}(\mathring{x})$ for all $i \in \{1, 2, \ldots, n\}$.

Definition 3.4. A point $\mathring{x}$ is a Pareto critical point for problem (3.1) if there exists no vector $\vartheta \in \mathbb{R}^m$ such that $\left[\nabla_{gH}\tilde{\mathcal{H}}_i(\mathring{x})^T\right]^{\alpha} \odot \vartheta \prec \tilde{0}^{\alpha}$ for all $i \in \{1, 2, \ldots, n\}$ and $\alpha \in [0, 1)$.

Example 3.5. Consider the bi-objective function $\tilde{\mathcal{H}}$ defined as follows

$$\begin{aligned}\tilde{\mathcal{H}}(x) &= (\tilde{\mathcal{H}}_1(x),\, \tilde{\mathcal{H}}_2(x)) \\ &= \left((1,2,3) \odot x_1^2 \oplus (2,3,4) \odot x_2^2\,,\ (1,3,5) \odot (x_1 - 1)^2 \oplus (1,2,3) \odot (x_2 - 1)^2\right).\end{aligned}$$

Let us compute the gradient of each of these fuzzy functions.

$$\nabla_{gH}\tilde{\mathcal{H}}_1(x) = \begin{pmatrix} (2,4,6) \odot x_1 \\ \\ (4,6,8) \odot x_2 \end{pmatrix} \quad \textit{and} \quad \nabla_{gH}\tilde{\mathcal{H}}_2(x) = \begin{pmatrix} (2,6,10) \odot (x_1 - 1) \\ \\ (2,4,6) \odot (x_2 - 1) \end{pmatrix}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_1^{\alpha}(x) = \begin{pmatrix} [2\alpha + 2,\ -2\alpha + 6] \odot x_1 \\ \\ [2\alpha + 4,\ -2\alpha + 8] \odot x_2 \end{pmatrix} \quad \textit{and}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_2^{\alpha}(x) = \begin{pmatrix} [4\alpha + 2,\ -4\alpha + 10] \odot (x_1 - 1) \\ \\ [2\alpha + 2,\ -2\alpha + 6] \odot (x_2 - 1). \end{pmatrix}$$

At the point $(0, 1)$, the set of descent directions of $\tilde{\mathcal{H}}_1$ is

$$\begin{aligned}\left\{\vartheta \in \mathbb{R}^2 \mid \left[\nabla\tilde{\mathcal{H}}_1(0,1)^T\right]^{\alpha} \odot \vartheta \prec \tilde{0}^{\alpha}\right\} &= \left\{(\vartheta_1, \vartheta_2) \in \mathbb{R}^2 \mid [2\alpha + 4,\ -2\alpha + 8] \odot \vartheta_2 \prec \tilde{0}^{\alpha}\right\} \\ &= \left\{(\vartheta_1, \vartheta_2) \in \mathbb{R}^2 \mid \vartheta_2 < 0,\ \forall\ \alpha \in [0,\, 1)\right\},\end{aligned}$$

and the set of descent directions of $\tilde{\mathcal{H}}_2$ is

$$\begin{aligned}\left\{\vartheta \in \mathbb{R}^2 \mid \left[\nabla\tilde{\mathcal{H}}_2(0,1)^T\right]^{\alpha} \odot \vartheta \prec \tilde{0}^{\alpha}\right\} &= \left\{(\vartheta_1, \vartheta_2) \in \mathbb{R}^2 \mid [4\alpha - 10,\ -4\alpha - 2] \odot \vartheta_1 \prec \tilde{0}^{\alpha}\right\} \\ &= \left\{(\vartheta_1, \vartheta_2) \in \mathbb{R}^2 \mid \vartheta_1 > 0,\ \forall\ \alpha \in [0,\, 1)\right\}.\end{aligned}$$

FIGURE 1. (a) shows that there exist common descent directions, such as $\vartheta = (1, -1)$, for both $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ such that $\left[\nabla_{gH}\tilde{\mathcal{H}}_i(0,1)^T\right]^{\alpha} \odot \vartheta \prec 0^{\alpha}$ for all $i \in \{1, 2\}$. Consequently, the point $(0, 1)$ is not a Pareto critical point.

At the point $\left(\frac{1}{2},\frac{1}{2}\right)$, the set of descent directions for $\tilde{\mathcal{H}}_1$ is

$$\begin{aligned}\left\{\vartheta \in \mathbb{R}^2 \mid \left[\nabla\tilde{\mathcal{H}}_1\left(\frac{1}{2},\frac{1}{2}\right)^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha\right\} &= \Big\{(\vartheta_1,\vartheta_2) \in \mathbb{R}^2 \mid [\alpha+1,\ -\alpha+3] \odot \vartheta_1 \\ &\quad \oplus [\alpha+2,\ -\alpha+4] \odot \vartheta_2 \prec \tilde{0}^\alpha\Big\} \\ &= \Big\{(\vartheta_1,\vartheta_2) \in \mathbb{R}^2 \mid (\alpha+1)\vartheta_1 \\ &\quad +(\alpha+2)\vartheta_2 < 0,\ (-\alpha+3)\vartheta_1 \\ &\quad +(-\alpha+4)\vartheta_2 < 0,\ (-\alpha+3)\vartheta_1 \\ &\quad +(\alpha+2)\vartheta_2 < 0,\ \ (\alpha+1)\vartheta_1 \\ &\quad +(-\alpha+4)\vartheta_2 < 0\Big\}.\end{aligned}$$

We observe that for all $\alpha \in [0,1)$, the region bounded by all these lines is the region bounded by the lines of the inequalities $(-\alpha+3)\vartheta_1 + (\alpha+2)\vartheta_2 < 0$ and $(\alpha+1)\vartheta_1 + (-\alpha+4)\vartheta_2 < 0$.
The set of descent directions of $\tilde{\mathcal{H}}_2$ is

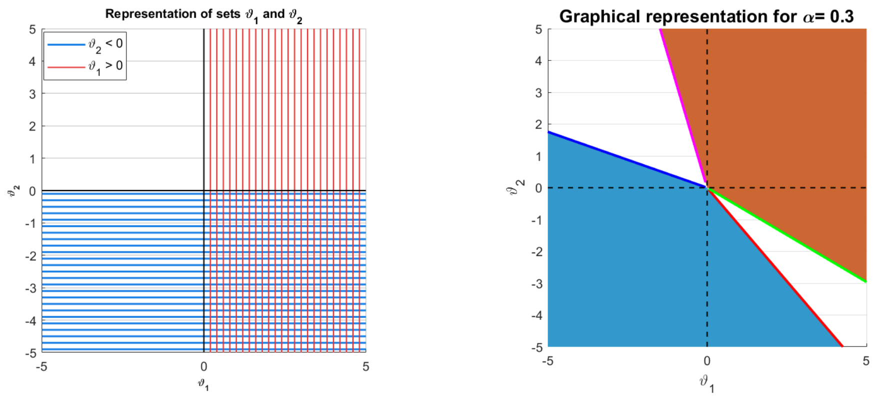
$$\begin{aligned}\left\{\vartheta \in \mathbb{R}^2 \mid \left[\nabla\tilde{\mathcal{H}}_2\left(\frac{1}{2},\frac{1}{2}\right)^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha\right\} &= \Big\{(\vartheta_1,\vartheta_2) \in \mathbb{R}^2 \mid [2\alpha-5,\ -2\alpha-1] \odot \vartheta_1 \\ &\quad \oplus [\alpha-3,\ -\alpha-1] \odot \vartheta_2 \prec \tilde{0}^\alpha\Big\} \\ &= \Big\{(\vartheta_1,\vartheta_2) \in \mathbb{R}^2 \mid (2\alpha-5)\vartheta_1 \\ &\quad +(\alpha-3)\vartheta_2 < 0,\ (-2\alpha-1)\vartheta_1 \\ &\quad +(-\alpha-1)\vartheta_2 < 0,\ (-2\alpha-1)\vartheta_1 \\ &\quad +(\alpha-3)\vartheta_2 < 0,\ (2\alpha-5)\vartheta_1 \\ &\quad +(-\alpha-1)\vartheta_2 < 0\Big\}.\end{aligned}$$

We observe that for all $\alpha \in [0,1)$ the region bounded by all these lines is the region bounded by the lines of the inequalities $(-2\alpha-1)\vartheta_1 + (\alpha-3)\vartheta_2 < 0$ and $(2\alpha-5)\vartheta_1 + (-\alpha-1)\vartheta_2 < 0$.

Let us graphically represent these sets.

From FIGURE 1. (b), we observe that for all $\alpha \in [0,1)$ there exists no descent direction $\vartheta \in \mathbb{R}^2$ common to both $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ such that $\left[\nabla_{gH}\tilde{\mathcal{H}}_i(\frac{1}{2},\frac{1}{2})^T\right]^\alpha \odot \vartheta \prec 0^\alpha$ for all $i \in \{1,2\}$. Consequently, the point $(\frac{1}{2},\frac{1}{2})$ is a Pareto critical point.
A natural question arises: what is the relationship between a weakly Pareto optimal point and a Pareto critical point? To address this, we establish the following results

Proposition 3.6. *If $\mathring{x}$ is a weakly Pareto optimal point of problem (3.1), then $\mathring{x}$ is a Pareto critical point of problem (3.1).*



(a) : Set of descent directions for $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ at the point (1, 0)

(b) : Set of descent directions for $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$ at the point (0.5, 0.5)

FIGURE 1. Geometric representation of descent directions at a Pareto and a non-Pareto point

Proof. Let $\mathring{x}$ be a weakly Pareto optimal point of problem (3.1). If $\mathring{x}$ is not a Pareto critical point of problem (3.1), then there exists $\vartheta \in \mathbb{R}^m$ such that:

$$\left[\nabla_{gH}\tilde{\mathcal{H}}_i(\mathring{x})^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha, \quad \forall\, i \in \{1,2,...,n\}.$$

From Lemma 2.9, we obtain

$$\lim_{\lambda\to 0}\frac{1}{\lambda}\odot\left[\tilde{\mathcal{H}}_i^\alpha(\mathring{x}+\lambda\vartheta)\ominus_{gH}\tilde{\mathcal{H}}_i^\alpha(\mathring{x})\right] = \left[\nabla_{gH}\tilde{\mathcal{H}}_i(\mathring{x})^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha, \quad \forall\, i \in \{1,2,...,n\}.$$

Consequently, there exists $\gamma > 0$ such that $\tilde{\mathcal{H}}_i^\alpha(\mathring{x}+\lambda\vartheta)\ominus_{gH}\tilde{\mathcal{H}}_i^\alpha(\mathring{x}) \prec \tilde{0}^\alpha$ for all $\lambda \in (0,\gamma)$ and for all $i \in \{1,2,\dots,n\}$. This implies that $\mathring{x}$ is not a weakly Pareto optimal point of problem (3.1), which contradicts the initial assumption. Therefore, $\mathring{x}$ is a Pareto critical point for problem (3.1). ☐

Remark 3.7. The converse of Proposition 3.6 is false.
Let us illustrate this with an example. Consider
$\tilde{\mathcal{H}}_1(x) = (0,2,5)\odot x$ and $\tilde{\mathcal{H}}_2(x) = (-4,-3,-1)\odot x^2$.
We have: $\nabla_{gH}\tilde{\mathcal{H}}_1(x) = (0,2,5)$ and $\nabla_{gH}\tilde{\mathcal{H}}_2(x) = (-8,-6,-2)\odot x$. For $x = 0$, there exists no $\vartheta \in \mathbb{R}^m$ such that $\nabla_{gH}\tilde{\mathcal{H}}_1(0)^T\odot\vartheta \prec 0$ and $\nabla_{gH}\tilde{\mathcal{H}}_2(0)^T\odot\vartheta \prec 0$. Hence, $x = 0$ is a Pareto critical point. However, $x = 0$ is not a weakly Pareto optimal point because we have
$\tilde{\mathcal{H}}_1(-1) = (-5,-2,0) \prec (0,0,0) = \tilde{\mathcal{H}}_1(0)$ and $\tilde{\mathcal{H}}_2(-1) = (-4,-3,-1) \prec (0,0,0) = \tilde{\mathcal{H}}_2(0)$.

The converse of Proposition 3.6 therefore holds only under the following condition

Proposition 3.8. *If for every $i \in \{1,2,\dots,n\}$, $\tilde{\mathcal{H}}_i$ is convex, then every Pareto critical point of problem (3.1) is a weakly Pareto optimal point for problem (3.1).*

Proof. Let $\mathring{x}$ be a Pareto critical point of problem (3.1). Then there exists an index $i_0 \in \{1, 2, \ldots, n\}$ such that $\left[\nabla_{gH}\tilde{\mathcal{H}}_{i_0}(\mathring{x})^T\right]^\alpha \odot \vartheta \not\prec 0$ for all $\vartheta \in \mathbb{R}^m$. Let $x^* \in \mathbb{R}^m$ and $\vartheta^* = x^* - \mathring{x}$. Then we have

$$\left[\nabla_{gH}\tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x})^T\right]^\alpha \odot (x^* - \mathring{x}) \not\prec \tilde{0}^\alpha. \tag{3.2}$$

Suppose that $\mathring{x}$ is not a weakly Pareto optimal point; then there exists a vector $x^* \in \mathbb{R}^m$ such that $\tilde{\mathcal{H}}^\alpha_{i_0}(x^*) \prec \tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x})$. Since $\tilde{\mathcal{H}}_{i_0}$ is a convex fuzzy function, then from Lemma 2.10 we have

$$\begin{aligned}
&\tilde{\mathcal{H}}^\alpha_{i_0}(x^*) \succeq \tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x}) \oplus \left[\nabla_{gH}\tilde{\mathcal{H}}_{i_0}(\mathring{x})^T\right]^\alpha \odot (x^* - \mathring{x}) \\
\Longrightarrow\quad &\tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x}) \succ \tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x}) \oplus \left[\nabla_{gH}\tilde{\mathcal{H}}_{i_0}(\mathring{x})^T\right]^\alpha \odot (x^* - \mathring{x}) \\
\Longrightarrow\quad &\left[\nabla_{gH}\tilde{\mathcal{H}}_{i_0}(\mathring{x})^T\right]^\alpha \odot (x^* - \mathring{x}) \prec \tilde{0}^\alpha.
\end{aligned}$$

This contradicts inequality (3.2). Consequently, there exists no vector $x^* \in \mathbb{R}^m$ such that
$\tilde{\mathcal{H}}_{i_0}(x^*) \prec \tilde{\mathcal{H}}_{i_0}(\mathring{x})$. Hence, for every $i \in \{1, 2, \ldots, n\}$, there exists no $x^* \in \mathbb{R}^m$ such that
$\tilde{\mathcal{H}}_i(x^*) \prec \tilde{\mathcal{H}}_i(\mathring{x})$. Therefore, $\mathring{x}$ is a weakly Pareto optimal point of problem (3.1). □

4. Gradient descent method for fuzzy multiobjective optimization

In this section, we develop a steepest descent method for finding Pareto critical points of the problem (3.1). We begin by computing a descent direction when the point is not critical. Next, we determine the step length to follow in this direction and, finally, present the complete algorithm of the method.

4.1. Descent direction.

A descent direction for a fuzzy multiobjective optimization problem is defined as follows:

Definition 4.1. A vector $\vartheta \in \mathbb{R}^m$ is a descent direction for problem (3.1) at a point $x \in \mathbb{R}^m$ if there exists $\gamma > 0$ such that

$$\tilde{\mathcal{H}}^\alpha_i(x + \lambda\vartheta) \prec \tilde{\mathcal{H}}^\alpha_i(x), \quad \forall\, \lambda \in (0, \gamma) \textit{ and } \forall\, i \in \{1, 2, \cdots, n\}.$$

Note that if, for every $i \in \{1, 2, \ldots, n\}$, $\tilde{\mathcal{H}}_i$ is a gH-differentiable fuzzy function and ϑ is a descent direction, then from Lemma 2.9 we have

$$\left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta = \lim_{\lambda\to 0} \frac{1}{\lambda} \odot \left[\tilde{\mathcal{H}}^\alpha_i(x + \lambda\vartheta) \ominus_{gH} \tilde{\mathcal{H}}^\alpha_i(x)\right] \prec \tilde{0}^\alpha, \quad \forall\, i \in \{1, 2, \cdots, n\}.$$

Hence, to find a descent direction $\vartheta \in \mathbb{R}^m$ at a point x for problem (3.1) amounts to finding a vector $\vartheta \in \mathbb{R}^m$ such that

$$\left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha, \quad \forall\, i \in \{1, 2, \cdots, n\}.$$

To identify such a vector $\vartheta \in \mathbb{R}^m$ at a point $\mathring{x}$, we define a fuzzy function $\tilde{h}^i_x : \mathcal{X} \to \mathfrak{F}$ for every $i \in \{1, 2, \ldots, n\}$ by

$$\tilde{h}^i_x(\vartheta) = \left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta. \tag{4.1}$$

If $\left[\tilde{\mathcal{H}}_i^L,\, \tilde{\mathcal{H}}_i^R\right]$ is the α-cut of the fuzzy function $\tilde{\mathcal{H}}_i$, the j-th component of $\nabla_{gH}\tilde{\mathcal{H}}_i(x)$ is given by

$$\left[\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j,\, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\right] = \left[\min\left\{\frac{\partial\tilde{\mathcal{H}}_i^L}{\partial x_j},\, \frac{\partial\tilde{\mathcal{H}}_i^R}{\partial x_j}\right\},\ \max\left\{\frac{\partial\tilde{\mathcal{H}}_i^L}{\partial x_j},\, \frac{\partial\tilde{\mathcal{H}}_i^R}{\partial x_j}\right\}\right].$$

Let $\tilde{h}_x^{i,L}$ and $\tilde{h}_x^{i,R}$ be, respectively, the lower and upper bounds of the fuzzy function $\tilde{h}_x^i$. Then, we have

$$\begin{cases} \tilde{h}_x^{i,L}(\vartheta) = \sum_{j=1}^{m} \min\left\{\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j,\, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right\}, \\ \\ \tilde{h}_x^{i,R}(\vartheta) = \sum_{j=1}^{m} \max\left\{\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j,\, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right\}. \end{cases} \tag{4.2}$$

Considering $|\vartheta| = (|\vartheta_1|,\, |\vartheta_2|, \ldots,\, |\vartheta_m|)^T$, we have

$$\begin{cases} \tilde{h}_x^{i,L}(\vartheta) & = \sum_{j=1}^{m}\frac{1}{2}\left[\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right) - |\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j|\right] \\ & = \frac{1}{2}\sum_{j=1}^{m}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\right)\vartheta_j - \frac{1}{2}\sum_{j=1}^{m}|\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j||\vartheta_j| \\ & = \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\right)^T\vartheta - \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\right)^T|\vartheta|, \\ \\ \tilde{h}_x^{i,R}(\vartheta) & = \sum_{j=1}^{m}\frac{1}{2}\left[\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right) + |\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j|\right] \\ & = \frac{1}{2}\sum_{j=1}^{m}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\right)\vartheta_j + \frac{1}{2}\sum_{j=1}^{m}|\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j||\vartheta_j| \\ & = \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\right)^T\vartheta + \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\right)^T|\vartheta|. \end{cases}$$

Consequently, we have

$$\begin{cases} \tilde{h}_x^{i,L}(\vartheta) = \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\right)^T\vartheta - \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\right)^T|\vartheta|, \\ \\ \tilde{h}_x^{i,R}(\vartheta) = \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\right)^T\vartheta + \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\right)^T|\vartheta|. \end{cases} \tag{4.3}$$

Proposition 4.2. *Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be the function defined by*

$$\varphi(\varpi) = \max_{i=1,2,\cdots,n}\varpi_i.$$

Then, φ is a Lipschitz function on $\mathbb{R}^n$ with Lipschitz constant equal to 1 for the norm $\|\cdot\|_\infty$.
Let $x \in \mathbb{R}^m$ be fixed and consider the function $\Psi_x : \mathbb{R}^m \to \mathbb{R}^n$ defined by

$$\Psi_x(\vartheta) = \left(\tilde{h}_x^{1,R}(\vartheta),\, \tilde{h}_x^{2,R}(\vartheta),\, \cdots,\, \tilde{h}_x^{n,R}(\vartheta)\right)^T,$$

where, for every $i \in \{1, 2, \ldots, n\}$, $\tilde{h}_x^{i,R}(\vartheta)$ denotes the upper bound of $\tilde{h}_x^i(\vartheta)$. Then, for every $\vartheta \in \mathbb{R}^m$, we have

$$(\varphi \circ \Psi_x)(\vartheta) = \max_{i \in \{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta). \tag{4.4}$$

Moreover, if each function $\tilde{h}_x^{i,R}$ is Lipschitz, then $\varphi \circ \Psi_x$ is also Lipschitz.

Proof. By the definition of φ, for every $\varpi = (\varpi_1, \ldots, \varpi_n)^T \in \mathbb{R}^n$, we have

$$\varphi(\varpi) = \max_{i=1,\ldots,n} \varpi_i.$$

It is well known that this function is Lipschitz with constant 1 with respect to the norm $\|\cdot\|_\infty$, because for all $\varpi, \varpi' \in \mathbb{R}^n$,

$$|\varphi(\varpi) - \varphi(\varpi')| \le \max_{i=1,\ldots,n} |\varpi_i - \varpi_i'| = \|\varpi - \varpi'\|_\infty.$$

For a fixed $x \in \mathbb{R}^m$, by the definition of Ψ_x,

$$(\varphi \circ \Psi_x)(\vartheta) = \varphi\big(\Psi_x(\vartheta)\big) = \varphi\Big(\tilde{h}_x^{1,R}(\vartheta), \ldots, \tilde{h}_x^{n,R}(\vartheta)\Big).$$

Using the explicit expression for φ, we obtain

$$(\varphi \circ \Psi_x)(\vartheta) = \max_{i=1,\ldots,n} \tilde{h}_x^{i,R}(\vartheta),$$

which establishes (4.4).

Finally, if each function $\tilde{h}_x^{i,R}$ is Lipschitz, then Ψ_x is Lipschitz, and since φ is Lipschitz, the composition $\varphi \circ \Psi_x$ is Lipschitz as a composition of Lipschitz functions. □

Remark 4.3. For a given $x \in \mathbb{R}^m$, we have: $\tilde{h}_x^{i,L}(\vartheta) \le \tilde{h}_x^{i,R}(\vartheta)$ for all $i \in \{1, 2, \ldots, n\}$. Hence, we obtain

$$\begin{aligned}
\varphi \circ \Psi_x(\vartheta) < 0 \quad &\Longrightarrow \max_{i \in \{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta) < 0 \\
&\Longrightarrow \left[\nabla_{gH} \tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \approx \tilde{h}_x^i(\vartheta) \prec \tilde{0}^\alpha, \quad \forall\, i \in \{1, 2, \cdots, n\}.
\end{aligned}$$

Since $\max_{i \in \{1,2,\ldots,n\}} \tilde{h}_x^{i,L}(\vartheta) < 0$ does not imply $\left[\nabla_{gH} \tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha$ for all $i \in \{1, 2, \ldots, n\}$, we shall use the upper bound $\tilde{h}_x^{i,R}(\vartheta)$ of $\tilde{h}_x^i(\vartheta)$ instead of $\tilde{h}_x^{i,L}(\vartheta)$ in (4.4).

Lemma 4.4. *$\varphi \circ \Psi_x : \mathbb{R}^m \to \mathbb{R}$ is a convex function.*

Proof. Let ω, $\vartheta \in \mathbb{R}^m$ and $\beta \in [0,\,1]$. We have

$$\begin{aligned}
\varphi\circ\Psi_x(\beta\omega+(1-\beta)\vartheta) &= \max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\beta\omega+(1-\beta)\vartheta)\\
&= \max_{i\in\{1,2,\cdots,n\}} \Big[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T(\beta\omega+(1-\beta)\vartheta)\\
&\qquad +\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)-\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T\,(|\beta\omega+(1-\beta)\vartheta|)\Big]\\
&\le \max_{i\in\{1,2,\cdots,n\}} \Big[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T(\beta\omega+(1-\beta)\vartheta)\\
&\qquad +\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)-\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T\,(\beta|\omega|+(1-\beta)|\vartheta|)\Big]\\
&\le \beta \max_{i\in\{1,2,\cdots,n\}} \Big[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T\omega\\
&\qquad +\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)-\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T|\omega|\Big]\\
&\qquad +(1-\beta) \max_{i\in\{1,2,\cdots,n\}} \Big[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T\vartheta\\
&\qquad +\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)-\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T|\vartheta|\Big]\\
&= \beta \max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\omega)+(1-\beta)\max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta)\\
&= \beta\;\varphi\circ\Psi_x(\omega)+(1-\beta)\;\varphi\circ\Psi_x(\vartheta).
\end{aligned}$$

Consequently, $\varphi\circ\Psi_x(\beta\omega+(1-\beta)\vartheta)\le\beta\,\varphi\circ\Psi_x(\omega)+(1-\beta)\,\varphi\circ\Psi_x(\vartheta)$, for all $\omega,\vartheta\in\mathbb{R}^m$ and for all $\beta\in[0,1]$. Hence, $\varphi\circ\Psi_x$ is a convex function. □

Lemma 4.5.
$\varphi\circ\Psi_x:\mathbb{R}^m\to\mathbb{R}$ is a positive homogeneous function.

Proof. Let $\omega\in\mathbb{R}^m$ and β a strictly positive real number. We have

$$\begin{aligned}
\varphi\circ\Psi_x(\beta\omega) &= \max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\beta\omega)\\
&= \max_{i\in\{1,2,\cdots,n\}} \Big[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T(\beta\omega)+\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\\
&\qquad -\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T|\beta\omega|\Big]\\
&= \beta \max_{i\in\{1,2,\cdots,n\}} \Big[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T(\omega)+\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\\
&\qquad -\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T|\omega|\Big]\\
&= \beta \max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\omega) \qquad (4.5)\\
&= \beta\;\varphi\circ\Psi_x(\omega).
\end{aligned}$$

Consequently, $\varphi \circ \Psi_x(\beta\omega) = \beta\, \varphi \circ \Psi_x(\omega)$ for all $\omega \in \mathbb{R}^m$ and for $\beta > 0$. This completes the proof. □

Let us now consider the following minimization problem

$$\min_{\vartheta \in \mathbb{R}^m} \left(\varphi \circ \Psi_x(\vartheta) + \frac{1}{2}\|\vartheta\|^2 \right). \tag{4.6}$$

By Lemma 4.4, the objective function of the minimization problem (4.6) is strongly convex. Consequently, problem (4.6) has a unique optimal solution.
Let $\vartheta(x)$ and $\zeta(x)$ be, respectively, the optimal solution and the optimal value of problem (4.6) at a point x. They are expressed as follows

$$\vartheta(x) = \arg\min_{\vartheta \in \mathbb{R}^m} \left(\varphi \circ \Psi_x(\vartheta) + \frac{1}{2}\|\vartheta\|^2 \right) \quad \textit{and} \quad \zeta(x) = \min_{\vartheta \in \mathbb{R}^m} \left(\varphi \circ \Psi_x(\vartheta) + \frac{1}{2}\|\vartheta\|^2 \right). \tag{4.7}$$

Remark 4.6. If $\tilde{\mathcal{H}}_x$ is a real-valued multiobjective function, then all the components $\tilde{\mathcal{H}}_x^{i,\,j}$ for $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$ are identical. Thus we have $\nabla_{gH}\tilde{\mathcal{H}}_x^i = \nabla_{gH}\tilde{\mathcal{H}}_x^{i,\,1} = \nabla_{gH}\tilde{\mathcal{H}}_x^{i,\,2} = \cdots = \nabla_{gH}\tilde{\mathcal{H}}_x^{i,\,j}$ for all $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Consequently, in such a particular case, the computation of $\vartheta(x)$ in (4.7) reduces to

$$\arg\min_{\vartheta \in \mathbb{R}^m} \left(\max_{i \in \{1,2,\cdots,n\}} (\nabla_{gH}\tilde{\mathcal{H}}_x^i)^T \odot \vartheta + \frac{1}{2}\|\vartheta\|^2 \right).$$

which is identical to the steepest descent direction for multiobjective optimization of real-valued functions.

To compute $\vartheta(x)$, we reformulate problem (4.6) as follows

$$\begin{aligned}
& \min_{\vartheta \in \mathbb{R}^m} \left(\max_{i \in \{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta) + \frac{1}{2}\|\vartheta\|^2 \right) \\
\equiv \quad & \min_{\vartheta \in \mathbb{R}^m} \Bigg(\max_{i \in \{1,2,\cdots,n\}} \Bigg[\frac{1}{2} \big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x) \big)^T \vartheta \\
& \qquad + \frac{1}{2} \big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x) \big)^T |\vartheta| \Bigg] + \frac{1}{2}\|\vartheta\|^2 \Bigg) \\
\equiv \quad & \min_{\omega,\, \vartheta \in \mathbb{R}^m} \Bigg(\max_{i \in \{1,2,\cdots,n\}} \Bigg[\frac{1}{2} \big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x) \big)^T \vartheta \\
& \qquad + \frac{1}{2} \big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x) \big)^T \omega \Bigg] + \frac{1}{2}\|\vartheta\|^2 \Bigg) \\
& \qquad \text{subject to} - \omega_j \leq \vartheta_j \leq \omega_j, \quad j \in \{1, 2, \cdots, m\}.
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
& \min_{\omega,\, \vartheta \in \mathbb{R}^m,\, \xi \in \mathbb{R}} \left(\xi + \frac{1}{2}\|\vartheta\|^2 \right) \\
& \text{S.t} \quad \begin{cases} \big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x) \big)^T \vartheta + \big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x) \big)^T \omega \leq 2\xi \\[2mm] -\omega_j \leq \vartheta_j \leq \omega_j, \quad j \in \{1, 2, \cdots, m\},\ i \in \{1, 2, \cdots, n\}. \end{cases}
\end{aligned} \tag{4.8}$$

Example 4.7. In this example, we consider the data from Example 3.5 and compute, as a function of α, the various steepest descent directions $\vartheta(x)$ at a point x that is not Pareto critical, using relation (4.8). From the objective function of Example 3.5, equation (4.8) can be reformulated as follows

(4.9)

$$\min_{\omega,\,\vartheta\in\mathbb{R}^m,\,\xi\in\mathbb{R}}\left(\xi+\frac{1}{2}\|\vartheta\|^2\right)$$
$$\text{S.t}\quad\begin{cases}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\right)^T\vartheta+\left(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)-\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\right)^T\omega\leq 2\xi\\[1em] \qquad\qquad -\omega_j\leq\vartheta_j\leq\omega_j,\quad j\in\{1,2\},\quad i\in\{1,2\}.\end{cases}$$

At the noncritical point (0, 1), (4.9) becomes

$$\min_{\omega,\,\vartheta\in\mathbb{R}^m,\,\xi\in\mathbb{R}}\left(\xi+\frac{1}{2}\|\vartheta\|^2\right)$$
$$\text{S.t}\quad\begin{cases}6\vartheta_2+2(1-\alpha)\omega_2\leq\xi\\ -6\vartheta_1+4(1-\alpha)\omega_1\leq\xi\\ -\omega_j\leq\vartheta_j\leq\omega_j,\quad j\in\{1,2\}.\end{cases}\tag{4.10}$$

By solving the quadratic programming problem (4.10), we obtain different solutions depending on α. These results are summarized in TABLE 1.

TABLE 1. Solutions to the problem of Example 3.5

α	0	0.1	0.2	0.3	0.4	0.5
ϑ	(1.6, −0.8)	(1.809, −1.033)	(1.993, −1.268)	(2.156, −1.5)	(2.304, −1.728)	(2.439, −1.951)

α	0.6	0.7	0.8	0.9	1
ϑ	(2.564, −2.169)	(2.681, −2.383)	(2.792, −2.592)	(2.898, −2.798)	(3, −3)

FIGURE 2 displays the various descent directions of problem (4.10).
We observe that each value of α yields a new descent direction within the region of descent directions common to both $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$.

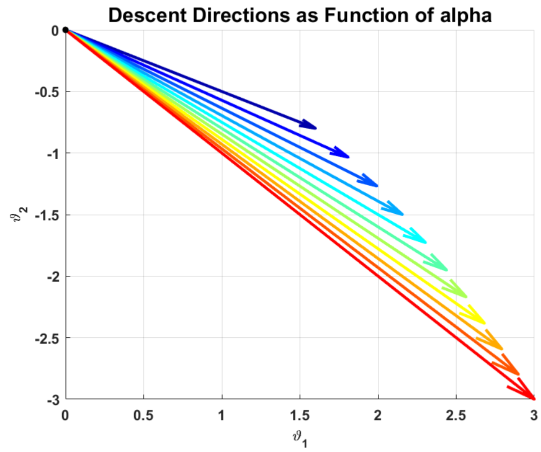


FIGURE 2. Descent direction as a function of α

Remark 4.8. If $n = 1$, then problem (3.1) is a single-objective optimization problem. In this case, the descent direction $\vartheta(x)$ is obtained by solving the following problem

$$\min_{\omega,\,\vartheta\in\mathbb{R}^m}\left(\max_{i\in\{1,2,\cdots,n\}}\left[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)+\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T\vartheta+\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x)-\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T\omega\right]\right.$$
$$\left.+\frac{1}{2}\|\vartheta\|^2\right)$$
$$\text{subject to} -\omega_j\le\vartheta_j\le\omega_j,\quad j\in\{1,2,\cdots,m\}.$$

For a single-objective real-valued function, we have $\nabla_{gH}\tilde{\mathcal{H}}^\alpha = \nabla_{gH}\tilde{\mathcal{H}}^L = \nabla_{gH}\tilde{\mathcal{H}}^R$. In this case, the descent direction computation reduces to

$$\arg\min_{\vartheta\in\mathbb{R}^m}\left(\max_{i\in\{1,2,\ldots,n\}}\left[\nabla_{gH}\tilde{\mathcal{H}}(x)^T\right]^\alpha\odot\vartheta+\frac{1}{2}\|\vartheta\|^2\right).$$

Note that the resulting descent direction in this case is $\vartheta = -\nabla_{gH}\tilde{\mathcal{H}}(x)$, which coincides with the descent direction for a single-objective real-valued optimization problem.

All descent directions $\vartheta(x)$ computed using (4.8) are linked to the notion of Pareto critical points of problem (3.1). The following results highlight this link as a stopping criterion for identifying critical points. These results state that if $\zeta(x) = 0$ and $\|\vartheta(x)\| = 0$, then x is a Pareto critical point. Otherwise, $\vartheta(x)$ is a descent direction for the objective functions of problem (3.1) at the point x.

Theorem 4.9. *Let $\vartheta(x)$ and $\zeta(x)$ be, respectively, the optimal solution and the optimal value of problem (4.6). Then the following results hold*

(i) $\zeta(x)\le 0$, *for all* $x\in\mathbb{R}^m$,
(ii) *If x is a critical point of problem (3.1), then $\vartheta(x) = 0$ and $\zeta(x) = 0$,*
(iii) *If x is not a Pareto critical point of problem (3.1), then $\zeta(x) < 0$ and $\vartheta(x)\neq 0$ is a descent direction for the objective functions of problem (3.1) at the point x,*

(iv) *The mapping* $x \longmapsto \vartheta(x)$ *is bounded on every compact subset of* $\mathbb{R}^m$,
(v) *The mapping* $x \longmapsto \zeta(x)$ *is continuous.*

Proof.

(i) For every $x \in \mathbb{R}^m$, we have

$$\begin{aligned}
&\zeta(x) \leq \varphi \circ \Psi_x(0) + \frac{1}{2}\|0\|^2 = \max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(0) + \frac{1}{2}\|0\|^2 \\
\Longrightarrow\quad &\zeta(x) \leq \max_{i\in\{1,2,\cdots,n\}} \Big[\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)\big)^T(0) + \frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x) \\
&\qquad -\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)\big)^T|0|\Big] \\
\Longrightarrow\quad &\zeta(x) \leq 0.
\end{aligned}$$

(ii) Let x be a Pareto critical point of problem (3.1). Then there exists $i_0 \in \{1,2,\ldots,n\}$ such that $\left[\nabla_{gH}\tilde{\mathcal{H}}_{i_0}(x)^T\right]^\alpha \odot \vartheta \nprec \tilde{0}^\alpha$ for every $\vartheta \in \mathbb{R}^m$. Consequently, for every $\vartheta \in \mathbb{R}^m$, we have

$$\begin{aligned}
\max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta) \geq 0 \qquad &\Longrightarrow \max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta) + \frac{1}{2}\|0\|^2 \geq 0 \\
&\Longrightarrow \max_{i\in\{1,2,\cdots,n\}} \varphi \circ \Psi_x(\vartheta) + \frac{1}{2}\|0\|^2 \geq 0 \\
&\Longrightarrow \zeta(x) = 0 \textit{ and } \vartheta(x) = 0.
\end{aligned}$$

(iii) Let x be a non-Pareto critical point of problem (3.1). Then there exists $\vartheta \in \mathbb{R}^m$ such that

$$\left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha, \quad \forall\, i \in \{1,2,\cdots,n\}.$$

Hence we obtain

$$\max_{i\in\{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta) < 0 \Longrightarrow \varphi \circ \Psi_x(\vartheta) < 0.$$

For every $\beta > 0$, we have

$$\zeta(x) \leq \varphi \circ \Psi_x(\beta\vartheta) + \frac{1}{2}\|\beta\vartheta\|^2.$$

Using Lemma 4.5, we have $\zeta(x) \leq \beta\left(\varphi \circ \Psi_x(\vartheta) + \frac{\beta}{2}\|\vartheta\|\right)$.
Take $0 < \beta < -\frac{2\varphi\circ\Psi_x(\vartheta)}{\|\vartheta\|^2}$. Then we obtain $\zeta(x) < 0$. Now $\zeta(x) < 0$ implies

$\vartheta(x) \neq 0$.

$$\begin{aligned}
\zeta(x) < 0 \qquad & \Longrightarrow \quad \varphi \circ \Psi_x(\vartheta) < -\frac{1}{2}\|\vartheta\|^2 \\
& \Longrightarrow \quad \max_{i \in \{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta) < 0 \\
& \Longrightarrow \quad \tilde{h}_x^{i,R}(\vartheta) < 0, \forall\ i \in \{1,2,\cdots,n\} \\
& \Longrightarrow \quad \tilde{h}_x^{i}(\vartheta) \prec \tilde{0}^{\alpha}, \quad \forall\ i \in \{1,2,\cdots,n\} \\
& \Longrightarrow \quad \left(\nabla_{gH}\tilde{\mathcal{H}}_i^{\alpha}(x)\right)^T \odot \vartheta \prec \tilde{0}^{\alpha}, \quad \forall\ i \in \{1,2,\cdots,n\}.
\end{aligned}$$

This implies that $\vartheta(x)$ is a descent direction for problem (3.1) at the point x.

(iv) Let $\mathfrak{R} \subset \mathbb{R}^m$ be a compact subset. For every $i \in \{1,2,\ldots,n\}$, we have

$$\begin{aligned}
& -\frac{1}{2}\|\nabla_{gH}\tilde{\mathcal{H}}_i^{L}(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^{R}(x)\|\|\vartheta\| + \frac{1}{2}\|\vartheta\|^2 \\
\leq\ & \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^{L}(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^{R}(x)\right)^T \vartheta + \frac{1}{2}\|\vartheta\|^2 \quad \text{(Cauchy - Schwarz inequality)} \\
\leq\ & \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^{L}(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^{R}(x)\right)^T \vartheta + \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^{R}(x) - \nabla_{gH}\tilde{\mathcal{H}}_i^{L}(x)\right)^T |\vartheta| + \frac{1}{2}\|\vartheta\|^2 \\
=\ & \tilde{h}_x^{i,R}(\vartheta) + \frac{1}{2}\|\vartheta\|^2 \\
\leq\ & \max_{i \in \{1,2,\cdots,n\}} \tilde{h}_x^{i,R}(\vartheta) + \frac{1}{2}\|\vartheta\|^2 \\
=\ & \varphi \circ \Psi_x(\vartheta) + \frac{1}{2}\|\vartheta\|^2 \\
=\ & \zeta(x) \\
\leq\ & 0.
\end{aligned}$$

Consequently, we obtain

$$\|\vartheta(x)\| \leq \|\nabla_{gH}\tilde{\mathcal{H}}_i^{L}(x) + \nabla_{gH}\tilde{\mathcal{H}}_i^{R}(x)\| \leq \|\nabla_{gH}\tilde{\mathcal{H}}_i^{L}(x)\| + \|\nabla_{gH}\tilde{\mathcal{H}}_i^{R}(x)\|. \tag{4.11}$$

Hence, there exists $N > 0$ such that

$$\|\vartheta(x)\| < N.$$

We deduce that the mapping $x \longrightarrow \vartheta(x)$ is bounded on every compact subset of $\mathbb{R}^m$.

(v) Let $\mathring{x}$ be an arbitrary point and $\{x^q\}$ a sequence such that $x^q \longrightarrow \mathring{x}$ as $q \longrightarrow +\infty$.
Let us show that $\lim_{q \to +\infty} \zeta(x^q) = \zeta(\mathring{x})$. From the optimality of $\vartheta(x^q)$, we have

for every q

$$\zeta(x^q) = \varphi \circ \Psi_{x^q}(\vartheta(x^q)) + \frac{1}{2}\|\vartheta(x^q)\|^2 \leq \varphi \circ \Psi_{x^q}(\vartheta(\mathring{x})) + \frac{1}{2}\|\vartheta(\mathring{x})\|^2$$

from equation (4.4) $\quad \zeta(x^q) \leq \max_{i\in\{1,2,\cdots,n\}} \tilde{h}^{i,R}_{x^q}(\vartheta(\mathring{x})) + \frac{1}{2}\|\vartheta(\mathring{x})\|^2$

from equation (4.3)
$$\zeta(x^q) \leq \max_{i\in\{1,2,\cdots,n\}} \Bigg(\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}^L_i(x^q) + \nabla_{gH}\tilde{\mathcal{H}}^R_i(x^q)\big)^T\vartheta(\mathring{x}) + \frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}^R_i(x^q) - \nabla_{gH}\tilde{\mathcal{H}}^L_i(x^q)\big)^T|\vartheta(\mathring{x})|\Bigg) + \frac{1}{2}\|\vartheta(\mathring{x})\|^2$$

Since $\nabla_{gH}\tilde{\mathcal{H}}^L_i$ and $\nabla_{gH}\tilde{\mathcal{H}}^R_i$ are continuous for every $i \in \{1,2,\ldots,n\}$ and $x^q \longrightarrow \mathring{x}$, we have

$$\nabla_{gH}\tilde{\mathcal{H}}^L_i(x^q) \longrightarrow \nabla_{gH}\tilde{\mathcal{H}}^L_i(\mathring{x})\, and\, \nabla_{gH}\tilde{\mathcal{H}}^R_i(x^q) \longrightarrow \nabla_{gH}\tilde{\mathcal{H}}^R_i(\mathring{x})\ when\ q \longrightarrow +\infty.$$

Consequently, we have

$$\lim_{q\to+\infty}\sup \zeta(x^q) \leq \max_{i\in\{1,2,\cdots,n\}} \Bigg(\frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}^L_i(\mathring{x}) + \nabla_{gH}\tilde{\mathcal{H}}^R_i(\mathring{x})\big)^T\vartheta(\mathring{x}) + \big(\nabla_{gH}\tilde{\mathcal{H}}^R_i(\mathring{x}) - \nabla_{gH}\tilde{\mathcal{H}}^L_i(\mathring{x})\big)^T|\vartheta(\mathring{x})|\Bigg) + \frac{1}{2}\|\vartheta(\mathring{x})\|^2$$

This implies

$$\lim_{q\to+\infty}\sup \zeta(x^q) \leq \varphi \circ \Psi_{\mathring{x}}(\vartheta(\mathring{x})) + \frac{1}{2}\|\vartheta(\mathring{x})\|^2 \tag{4.12}$$
$$\leq \zeta(\mathring{x}). \tag{4.13}$$

Moreover, we have

$$\zeta(\mathring{x}) = \min_{\vartheta\to\mathbb{R}^m}\left(\varphi \circ \Psi_{\mathring{x}}(\vartheta(\mathring{x})) + \frac{1}{2}\|\vartheta(\mathring{x})\|^2\right) \leq \varphi \circ \Psi_{\mathring{x}}(\vartheta(x^q)) + \frac{1}{2}\|\vartheta(x^q)\|^2.$$

Consequently, we have

$$\begin{aligned}\zeta(\mathring{x}) &\leq \lim_{q\to+\infty}\inf\left(\varphi \circ \Psi_{\mathring{x}}(\vartheta(x^q)) + \frac{1}{2}\|\vartheta(x^q)\|^2\right)\\ &= \lim_{q\to+\infty}\inf\left[\varphi \circ \Psi_{\mathring{x}}(\vartheta(x^q)) + \varphi \circ \Psi_{x^q}(\vartheta(x^q)) - \varphi \circ \Psi_{x^q}(\vartheta(x^q)) + \frac{1}{2}\|\vartheta(x^q)\|^2\right]\\ &= \lim_{q\to+\infty}\inf\Big[\zeta(x^q) + \varphi \circ \Psi_{\mathring{x}}(\vartheta(x^q)) - \varphi \circ \Psi_{x^q}(\vartheta(x^q))\Big].\end{aligned}$$

Since φ is a Lipschitz continuous function with Lipschitz constant 1, we have:

$$\zeta(\mathring{x}) \leq \Big[\zeta(x^q) + \|\Psi_{x^q}(\vartheta(x^q)) - \Psi_{\mathring{x}}(\vartheta(x^q))\|\Big].$$

we have

$$\Psi_{x^q}(\vartheta(x^q)) = \Big(\tilde{h}^{1,R}_{x^q}(\vartheta(x^q)),\, \tilde{h}^{2,R}_{x^q}(\vartheta(x^q)),\, \cdots,\, \tilde{h}^{n,R}_{x^q}(\vartheta(x^q))\Big)^T$$

and

$$\Psi_{\mathring{x}}(\vartheta(x^q)) = \left(\tilde{h}^{1,R}_{\mathring{x}}(\vartheta(x^q)),\, \tilde{h}^{2,R}_{\mathring{x}}(\vartheta(x^q)),\, \cdots,\, \tilde{h}^{n,R}_{\mathring{x}}(\vartheta(x^q))\right)^T.$$

From equation (4.3), for all $i \in \{1, 2, \cdots, n\}$, we have

$$\begin{aligned}\tilde{h}^{1,R}_{x^q}(\vartheta(x^q)) \qquad &= \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}^L_i(x^q) + \nabla_{gH}\tilde{\mathcal{H}}^R_i(x^q)\right)^T\vartheta(x^q) + \left(\nabla_{gH}\tilde{\mathcal{H}}^R_i(x^q)\right.\\ &\qquad \left.-\nabla_{gH}\tilde{\mathcal{H}}^L_i(x^q)\right)^T|\vartheta(x^q)|\end{aligned}$$

and

$$\begin{aligned}\tilde{h}^{1,R}_{\mathring{x}}(\vartheta(x^q)) \qquad &= \frac{1}{2}\big(\nabla_{gH}\tilde{\mathcal{H}}^L_i(\mathring{x}) + \nabla_{gH}\tilde{\mathcal{H}}^R_i(\mathring{x})\big)^T\omega(x^q) + \big(\nabla_{gH}\tilde{\mathcal{H}}^R_i(\mathring{x})\\ &\qquad -\nabla_{gH}\tilde{\mathcal{H}}^L_i(\mathring{x})\big)^T|\vartheta(x^q)|.\end{aligned}$$

From equation (4.11), we have $\vartheta(x) \le \|\nabla_{gH}\tilde{\mathcal{H}}^L_i(x)\| + \|\nabla_{gH}\tilde{\mathcal{H}}^R_i(x)\|$. Since $\nabla_{gH}\tilde{\mathcal{H}}^L_i(x)$ and $\nabla_{gH}\tilde{\mathcal{H}}^R_i(x)$ are continuous and $x^q \longrightarrow \mathring{x}$ as $q \longrightarrow +\infty$, we conclude that $\{\vartheta(x^q)\}$ is bounded. Then for $q \longrightarrow +\infty$, we have $\tilde{h}^{1,R}_{x^q}(\vartheta(x^q)) - \tilde{h}^{1,R}_{\mathring{x}}(\vartheta(x^q)) \longrightarrow 0$.
Consequently, $\Psi_{x^q}(\vartheta(x^q)) - \Psi_{\mathring{x}}(\vartheta(x^q)) \longrightarrow 0$ as $q \longrightarrow +\infty$. We then obtain

$$\zeta(x) \le \lim_{q\to+\infty}\inf \zeta(x^q). \tag{4.14}$$

From equations (4.12) and (4.14), we obtain $\lim\limits_{q\to+\infty}\sup \zeta(x^q) \le \zeta(\mathring{x}) \le \lim\limits_{q\to+\infty}\inf \zeta(x^q)$. This implies that ζ is continuous.

□

4.2. **Step length.** Assume that
$\vartheta \in \mathbb{R}^m$ is a direction vector such that $\left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha$ for all $i \in \{1, 2, \ldots, n\}$. The step length $s > 0$ is determined using the Armijo line search rule. Given a constant $\alpha \in [0, 1)$, a step s is acceptable if it satisfies

$$\begin{aligned}\tilde{\mathcal{H}}^\alpha_i(x + s\vartheta) \preceq &\tilde{\mathcal{H}}^\alpha_i(x) + [\theta s,\, \theta s] \odot \sum_{j=1}^m \max\left[\nabla_{gH}\tilde{\mathcal{H}}^L_i(x)_j\omega_j,\, \nabla_{gH}\tilde{\mathcal{H}}^R_i(x)_j\omega_j\right],\\ &\forall\, i \in \{1, 2, \cdots, n\}.\end{aligned} \tag{4.15}$$

For greater precision, the acceptance condition for s is

$$\begin{cases}\tilde{\mathcal{H}}^L_i(x + s\vartheta) \preceq \tilde{\mathcal{H}}^L_i(x) + \theta s \displaystyle\sum_{j=1}^m \max\left(\nabla_{gH}\tilde{\mathcal{H}}^L_i(x)_j\vartheta_j,\, \nabla_{gH}\tilde{\mathcal{H}}^R_i(x)_j\vartheta_j\right),\\[2ex] \tilde{\mathcal{H}}^R_i(x + s\vartheta) \preceq \tilde{\mathcal{H}}^R_i(x) + \theta s \displaystyle\sum_{j=1}^m \max\left(\nabla_{gH}\tilde{\mathcal{H}}^L_i(x)_j\vartheta_j,\, \nabla_{gH}\tilde{\mathcal{H}}^R_i(x)_j\vartheta_j\right).\end{cases} \tag{4.16}$$

To compute s, we initialize $s = 1$. While conditions (4.16) are not satisfied, we reduce s by setting $s = \eta s$, where $\eta \in (0, 1)$ is the reduction factor. The guarantee of the existence of such a step is established in the following result.

Theorem 4.10. *Let $\tilde{\mathcal{H}}_i : \mathbb{R}^m \longrightarrow \mathfrak{F}$ be a gH-continuously differentiable fuzzy function for every*
$i \in \{1, 2, \ldots, n\}$ and let $\vartheta \in \mathbb{R}^m$. If $\left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \omega \prec \tilde{0}^\alpha$ for all $i \in \{1, 2, \ldots, n\}$ and $\theta \in (0, 1)$, then there exists $\gamma > 0$ such that for every $i \in \{1, 2, \ldots, n\}$,

$$\tilde{\mathcal{H}}_i^\alpha(x+s\vartheta) \preceq \tilde{\mathcal{H}}_i^\alpha(x)+[\theta s,\, \theta s]\odot\sum_{j=1}^m \max\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j,\, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right), \quad \forall\, \theta \in (0, \gamma]. \tag{4.17}$$

Proof. Since $\tilde{\mathcal{H}}_i : \mathbb{R}^m \longrightarrow \mathfrak{F}$ is a gH-continuously differentiable fuzzy function for every
$i \in \{1, 2, \ldots, n\}$, $\left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \prec \tilde{0}^\alpha$, and $\theta \in (0, 1)$, then by Lemma 2.9, we have

$$\lim_{s\to 0}\frac{1}{s}\left[\tilde{\mathcal{H}}_i^\alpha(x + s\vartheta) \ominus_{gH} \tilde{\mathcal{H}}_i^\alpha(x)\right] = \left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \prec \theta \odot \left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta,$$
$$\forall\; i \in \{1, 2, \cdots, n\}.$$

Consequently, there exists $\gamma > 0$ such that for every $s \in (0, \gamma]$ and for every $i \in \{1, 2, \ldots, n\}$, we have

$$\begin{aligned} &\tilde{\mathcal{H}}_i^\alpha(x + s\vartheta) \ominus_{gH} \tilde{\mathcal{H}}_i^\alpha(x) \prec \theta s \odot \left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \\ &\Longrightarrow \tilde{\mathcal{H}}_i^\alpha(x + s\vartheta) \prec \tilde{\mathcal{H}}_i^\alpha(x) \oplus \theta s \odot \left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \end{aligned} \tag{4.18}$$

Note that

$$\begin{aligned} \left[\nabla_{gH}\tilde{\mathcal{H}}_i(x)^T\right]^\alpha \odot \vartheta \quad &= \Bigg[\sum_{j=1}^m \min\left\{\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right\}, \\ &\qquad \sum_{j=1}^m \max\left\{\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right\}\Bigg] \\ &\preceq \Bigg[\sum_{j=1}^m \max\left\{\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right\}, \\ &\qquad \sum_{j=1}^m \max\left\{\nabla_{gH}\tilde{\mathcal{H}}_i^L(x)_j\vartheta_j, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x)_j\vartheta_j\right\}\Bigg]. \end{aligned}$$

which completes the proof. □

We now present the algorithm of the steepest descent method for identifying the Pareto critical points of problem (3.1).

Algorithm 1 Steepest descent method for solving the problem (3.1)

1: Input the fuzzy multi-objective function $\tilde{\mathcal{H}} = (\tilde{\mathcal{H}}_1, \tilde{\mathcal{H}}_2, \ldots, \tilde{\mathcal{H}}_n)^T$, where each $\tilde{\mathcal{H}}_i$,
$i \in \{1, 2, \ldots, n\}$, is a gH-continuously differentiable fuzzy function. Choose a tolerance $\varepsilon > 0$, Maximum number of iterations i_{max} and initialize $q = 0$;
2: While $q < i_{max}$, do
3: Determine $\tilde{\mathcal{H}}_i^\alpha$ for all $i \in \{1, 2, \ldots, n\}$;
4: Choose the step reduction factor $\eta \in (0, 1)$ and a line search parameter $\theta \in (0, 1)$ for the Armijo rule (4.16);
5: Compute $\left[\nabla_{gH} \tilde{\mathcal{H}}_i(x^q)\right]^\alpha$ for all $i \in \{1, 2, \ldots, n\}$;
6: Compute the optimal solution $\vartheta(x^q)$ and the optimal value $\zeta(x^q)$ of problem (4.6) using (4.7);
7: If $\zeta(x^q) > -\varepsilon$, return x^q as a Pareto critical point. Otherwise, go to Step 8;
8: Initialize $s_q \leftarrow 1$. Then reduce the step length by $s_q = \eta s_q$ until (4.16) is satisfied;
9: Update $x^{q+1} = x^q + s_q \vartheta(x^q)$, increment q, and restart the process from Step 3.

Since the objective function of problem (4.6) is strongly convex, it admits a unique optimal solution. This guarantees the existence of the descent direction $\vartheta(x^q)$ and consequently ensures that Step 6 is well-defined. Moreover, Theorem 4.10 establishes the existence of a step length s_q, which guarantees that Step 8 is well-defined. Thus Algorithm 1 is well-posed.

5. Performance analysis of the method

To assess the performance of the proposed method, we present in this section a study of the convergence and complexity of Algorithm 1.

5.1. Convergence analysis.

Regarding the convergence analysis, if the algorithm stops after a finite number of iterations, then Step 5 clearly indicates that the obtained result is a Pareto critical point. Otherwise, we assume that starting from an initial point x^0, the algorithm generates an infinite sequence $\{x^q\}$ such that $\zeta(x^q) \neq 0$ for $q \in \mathbb{N}$. The following result shows that all limit points of subsequences of x^q are Pareto critical points of the fuzzy multiobjective optimization problem (3.1).

Theorem 5.1. *Every accumulation point of the sequence $\{x^q\}$ generated by Algorithm 1 is a Pareto critical point of the fuzzy multiobjective optimization problem (3.1). Moreover, if the set*

$$P_0 = \left\{ x \in \mathbb{R}^m : \tilde{\mathcal{H}}_i^\alpha(x) \preceq \tilde{\mathcal{H}}_i^\alpha(x^0), \quad i \in \{1, 2, \cdots, n\} \right\} \tag{5.1}$$

is bounded, then the sequence $\{x^q\}$ remains bounded and possesses at least one accumulation point.

Proof. Let $\mathring{x}$ be an accumulation point of the sequence $\{x^q\}$. We show that $\mathring{x}$ is a Pareto critical point of the fuzzy multiobjective optimization problem (3.1). Let $\vartheta(\mathring{x})$ and $\zeta(\mathring{x})$ be, respectively, the optimal solution and the optimal value of the

minimization problem (4.6) at the point $\mathring{x}$.
By Theorem 4.9, $\mathring{x}$ is a Pareto critical point of problem (3.1) if $\zeta(\mathring{x}) = 0$. Let us show that $\zeta(\mathring{x}) = 0$. For every $q \in \mathbb{N}$ and for every $i \in \{1, 2, \ldots, n\}$, we have $\tilde{\mathcal{H}}_i^\alpha(x^{q+1}) \preceq \tilde{\mathcal{H}}_i^\alpha(x^q)$ and
$\lim\limits_{q\to+\infty} \tilde{\mathcal{H}}_i^\alpha(x^q) \approx \tilde{\mathcal{H}}_i^\alpha(\mathring{x})$. Consequently, we obtain

$$\begin{aligned}
&\lim_{q\to+\infty} \tilde{\mathcal{H}}_i^L(x^q) = \tilde{\mathcal{H}}_i^L(\mathring{x}) \quad \text{and} \quad \lim_{q\to+\infty} \tilde{\mathcal{H}}_i^R(x^q) = \tilde{\mathcal{H}}_i^R(\mathring{x}) \\
\Longrightarrow \quad &\lim_{q\to+\infty} |\tilde{\mathcal{H}}_i^L(x^{q+1}) - \tilde{\mathcal{H}}_i^L(x^q)| = 0 \quad \text{and} \quad \lim_{q\to+\infty} |\tilde{\mathcal{H}}_i^R(x^{q+1}) - \tilde{\mathcal{H}}_i^R(x^q)| = 0.
\end{aligned}$$

This implies that

$$\lim_{q\to+\infty} \|\tilde{\mathcal{H}}_i^\alpha(x^{q+1}) \ominus_{gH} \tilde{\mathcal{H}}_i^\alpha(x^q)\|_{\mathfrak{F}} = 0. \tag{5.2}$$

Since $\tilde{\mathcal{H}}_i^\alpha(x^q) \ominus_{gH} \tilde{\mathcal{H}}_i^\alpha(x^{q+1}) \succeq (-1) \odot \theta s_q \odot \left[\nabla_{gH} \tilde{\mathcal{H}}_i(x^q)^T\right]^\alpha \odot \vartheta(x^q) \succeq \tilde{0}^\alpha$ for every $i \in \{1, 2, \ldots, n\}$, we obtain from equation (5.2)

$$\lim_{q\to+\infty} s_q \odot \left[\nabla_{gH} \tilde{\mathcal{H}}_i(x^q)^T\right]^\alpha \odot \vartheta(x^q) \approx \tilde{0}^\alpha \text{ for all } i \in \{1, 2, \cdots, n\}.$$

Since $s_q \in (0, 1]$ for every $q \in \mathbb{N}$, we have the following two cases: $\limsup_{q\to+\infty} s_q > 0$ or $\limsup_{q\to+\infty} s_q = 0$.
Case 1 : Assume $\limsup_{q\to+\infty} s_q > 0$. Then there exists a subsequence $\{x^{q_p}\}$ converging to $\mathring{x}$ and a $\mathring{p} > 0$ such that $\lim\limits_{p\to+\infty} s_{q_p} = \mathring{x}$. Therefore, we obtain

$$\begin{aligned}
&\lim_{p\to+\infty} \left[\nabla_{gH} \tilde{\mathcal{H}}_i(x^{q_p})^T\right]^\alpha \odot \vartheta(x^{q_p}) \approx \tilde{0}^\alpha \\
(4.1) \Longrightarrow\ &\lim_{p\to+\infty} \tilde{h}^i_{x^{q_p}}(\vartheta(x^{q_p})) = \tilde{0}^\alpha \\
\Longrightarrow\ &\lim_{p\to+\infty} \left[\tilde{h}^{i,L}_{x^{q_p}}(\vartheta(x^{q_p})), \tilde{h}^{i,R}_{x^{q_p}}(\vartheta(x^{q_p}))\right] = \tilde{0}^\alpha \\
\Longrightarrow\ &\lim_{p\to+\infty} \tilde{h}^{i,R}_{x^{q_p}}(\vartheta(x^{q_p})) = \tilde{0}^\alpha \\
\Longrightarrow\ &\lim_{p\to+\infty} \left(\tilde{h}^{i,R}_{x^{q_p}}(\vartheta(x^{q_p})) + \frac{1}{2}\|\vartheta(x^{q_p})\|^2\right) \geq 0 \\
\Longrightarrow\ &\lim_{p\to+\infty} \left(\max_{i\in\{1,2,\cdots,n\}} \tilde{h}^{i,R}_{x^{q_p}}(\vartheta(x^{q_p})) + \frac{1}{2}\|\vartheta(x^{q_p})\|^2\right) \geq 0 \\
(4.4) \Longrightarrow\ &\lim_{p\to+\infty} \left(\varphi \circ \Psi_{x^{q_p}}(\vartheta(x^{q_p})) + \frac{1}{2}\|\omega(x^{q_p})\|^2\right) \geq 0 \\
\Longrightarrow\ &\lim_{p\to+\infty} \zeta(x^{q_p}) \geq 0 \\
\Longrightarrow\ &\zeta(\mathring{x}) \geq 0, \quad \text{because by Theorem 4.9, } \zeta \text{ is continuous} \\
\Longrightarrow\ &\zeta(\mathring{x}) = 0 \quad \text{since } \zeta(x) \leq 0\ \forall\ x \in \mathbb{R}^m.
\end{aligned}$$

Then by Theorem 4.9, $\mathring{x}$ is a Pareto critical point for the problem (3.1).
Case 2 : Assume $\limsup_{q\to 0} s_q = 0$. From equation (4.11), we have

$$\|\vartheta(x)\| \leq \|\nabla_{gH} \tilde{\mathcal{H}}_i^L(x^q)\| + \|\nabla_{gH} \tilde{\mathcal{H}}_i^R(x^q)\|. \tag{5.3}$$

Since $\nabla_{gH}\tilde{\mathcal{H}}_i^L$ and $\nabla_{gH}\tilde{\mathcal{H}}_i^R$ are continuous functions and $\lim\limits_{q\to+\infty} x^q = \mathring{x}$, the sequence $\{\vartheta(x^q)\}$ is bounded. Consequently, $\{\vartheta(x^q)\}$ contains a convergent subsequence. Because $\lim\limits_{q\to+\infty} x^q = \mathring{x}$, $\limsup_{q\to 0} s_q = 0$, and the sequence $\{\vartheta(x^q)\}$ possesses a convergent subsequence, we may consider the subsequences $\{x^{q_p}\}$, $\{\vartheta(x^{q_p})\}$, and $\{s_{q_p}\}$ converging respectively to $\mathring{x}$, $\mathring{\vartheta}$, and 0. For every $p \in \mathbb{N}$, we have

(5.4)
$$\max_{i\in\{1,2,\cdots,n\}} \tilde{h}^{i,R}_{x^{q_p}}(\vartheta(x^{q_p})) = \varphi\circ\Psi_{x^{q_p}}(\vartheta(x^{q_p})) \leq \varphi\circ\Psi_{x^{q_p}}(\vartheta(x^{q_p})) + \frac{1}{2}\|\vartheta(x^{q_p})\|^2 = \zeta(x^{q_p}) \leq 0.$$

From equation (4.3), we have

$$\begin{aligned}\tilde{h}^{i,R}_{x^{q_p}}(\vartheta(x^{q_p})) =& \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x^{q_p}) + \nabla_{gH}\tilde{\mathcal{H}}_i^R(x^{q_p})\right)^T \vartheta(x^{q_p}) \\ &+ \frac{1}{2}\left(\nabla_{gH}\tilde{\mathcal{H}}_i^R(x^{q_p}) - \nabla_{gH}\tilde{\mathcal{H}}_i^L(x^{q_p})\right)^T |\vartheta(x^{q_p})|.\end{aligned}$$

Hence $\tilde{h}^{i,R}_{x^{q_p}}(\vartheta(x^{q_p})) \longrightarrow \tilde{h}^{i,R}_{\mathring{x}}(\mathring{\vartheta})$ as $p \longrightarrow +\infty$. Consequently, from equation (5.4), we have:

$$\max_{i\in\{1,2,\cdots,n\}} \tilde{h}^{i,R}_{\mathring{x}}(\mathring{\vartheta}) \leq \zeta(\mathring{x}) \leq 0 \tag{5.5}$$

Consider an arbitrary fixed positive integer ι. Since $s_{q_p} \longrightarrow 0$ as $p \longrightarrow +\infty$ and $\eta \in (0,1)$, we have $s_{q_p} \leq \eta^\iota$, which means that equation (4.16) is not satisfied at x^{q_p} for $t = \eta^\iota$. Hence, for every p, there exists $i(q_p) \in \{1,2,\dots,n\}$ such that one of the following inequalities holds

(5.6)
$$\begin{cases} \tilde{\mathcal{H}}_i^L(x^{q_p} + \eta^\iota\vartheta(x^{q_p})) > \tilde{\mathcal{H}}_i^L(x^{q_p}) \\ \qquad + \theta\eta^\iota \displaystyle\sum_{j=1}^m \max\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x^{q_p})_j\vartheta(x^{q_p})_j,\, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x^{q_p})_j\vartheta(x^{q_p})_j\right), \\ \tilde{\mathcal{H}}_i^R(x^{q_p} + \eta^\iota\vartheta(x^{q_p})) > \tilde{\mathcal{H}}_i^R(x^{q_p}) \\ \qquad + \theta\eta^\iota \displaystyle\sum_{j=1}^m \max\left(\nabla_{gH}\tilde{\mathcal{H}}_i^L(x^{q_p})_j\vartheta(x^{q_p})_j,\, \nabla_{gH}\tilde{\mathcal{H}}_i^R(x^{q_p})_j\vartheta(x^{q_p})_j\right). \end{cases}$$

Since $i(q_p) \subset \{1,2,\dots,n\}$, there exists a subsequence $\{q_{p_r}\}_r$ and an index i_0 such that
$i_0 = i(q_{p_r})$ for every $r \in \{1,2,\dots,n\}$, and which satisfies one of the following inequalities

(5.7)
$$\begin{cases} \tilde{\mathcal{H}}_{i_0}^\alpha(x^{q_{p_r}} + \eta^\iota\vartheta(x^{q_{p_r}})) > \tilde{\mathcal{H}}_{i_0}^L(x^{q_{p_r}}) \\ \qquad + \theta\eta^\iota \displaystyle\sum_{j=1}^m \max\left(\nabla_{gH}\tilde{\mathcal{H}}_{i_0}^L(x^{q_{p_r}})_j\vartheta(x^{q_{p_r}})_j,\, \nabla_{gH}\tilde{\mathcal{H}}_{i_0}^R(x^{q_{p_r}})_j\vartheta(x^{q_{p_r}})_j\right), \\ \tilde{\mathcal{H}}_{i_0}^R(x^{q_{p_r}} + \eta^\iota\vartheta(x^{q_{p_r}})) > \tilde{\mathcal{H}}_{i_0}^R(x^{q_{p_r}}) \\ \qquad + \theta\eta^\iota \displaystyle\sum_{j=1}^m \max\left(\nabla_{gH}\tilde{\mathcal{H}}_{i_0}^L(x^{q_{p_r}})_j\vartheta(x^{q_{p_r}})_j,\, \nabla_{gH}\tilde{\mathcal{H}}_{i_0}^R(x^{q_{p_r}})_j\vartheta(x^{q_{p_r}})_j\right). \end{cases}$$

Taking $r \longrightarrow +\infty$, we obtain

$$(5.8)\quad \begin{cases} \tilde{\mathcal{H}}^L_{i_0}(\mathring{x}+\eta^\iota\mathring{\vartheta})) \geq \tilde{\mathcal{H}}^L_{i_0}(\mathring{x}) + \theta\eta^\iota \sum_{j=1}^{m} \max\left(\nabla_{gH}\tilde{\mathcal{H}}^L_{i_0}(\mathring{x})_j\mathring{\vartheta}_j,\, \nabla_{gH}\tilde{\mathcal{H}}^R_{i_0}(\mathring{x})_j\mathring{\vartheta}_j\right), \\ \\ \tilde{\mathcal{H}}^R_{i_0}(\mathring{x}+\eta^\iota\mathring{\vartheta})) \geq \tilde{\mathcal{H}}^R_{i_0}(\mathring{x}) + \theta\eta^\iota \sum_{j=1}^{m} \max\left(\nabla_{gH}\tilde{\mathcal{H}}^L_{i_0}(\mathring{x})_j\mathring{\vartheta}_j,\, \nabla_{gH}\tilde{\mathcal{H}}^R_{i_0}(\mathring{x})_j\mathring{\vartheta}_j\right). \end{cases}$$

Hence, we obtain

$$\tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x}+\eta^\iota\mathring{\vartheta})) \not\prec \tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x}) \oplus [\theta\eta^\iota,\, \theta\eta^\iota] \sum_{j=1}^{m} \max\left(\nabla_{gH}\tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x})_j\mathring{\vartheta}_j,\, \nabla_{gH}\tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x})_j\mathring{\vartheta}_j\right)$$

This holds for every positive integer ι. Therefore, by Theorem 4.10, we have: $\left(\nabla_{gH}\tilde{\mathcal{H}}^\alpha_{i_0}(\mathring{x})\right)\odot \mathring{\vartheta} \not\prec \tilde{0}^\alpha$, which implies $\tilde{h}^{i_0,R}_{\mathring{x}}(\mathring{\vartheta}) \geq 0$. Thus, we obtain

$$(5.9)\qquad \max_{i\in\{1,2,\cdots,n\}} \tilde{h}^{i,R}_{\mathring{x}}(\mathring{\vartheta}) \geq 0$$

Using equations (5.5) and (5.9), we have $\zeta(\mathring{x}) = 0$. Consequently, by Theorem 4.9, $\mathring{x}$ is a Pareto critical point of problem (3.1).
Since for every $q \in \mathbb{N}$ and for all $i \in \{1, 2, \ldots, n\}$, $\tilde{\mathcal{H}}^\alpha_i(x^{q+1}) \preceq \tilde{\mathcal{H}}^\alpha_i(x^q)$, the sequence $\{x^q\}$ is contained in the bounded level set

$$P_0 = \left\{x \in \mathbb{R}^m : \tilde{\mathcal{H}}^\alpha_i(x) \preceq \tilde{\mathcal{H}}^\alpha_i(x^0), \quad i \in \{1, 2, \cdots, m\}\right\}.$$

Thus, the sequence $\{x^q\}$ is bounded and possesses at least one accumulation point, which completes the proof. □

5.2. **Complexity analysis.** This section is devoted to the study of the complexity of the proposed method. In this study, let m denote the number of decision variables, n the number of objective functions, σ the optimal reduction factor and M the maximum number of iterations.

Let us analyze the complexity of the method at each step.

1. Initialization of the decision variables has complexity $O(mn)$ because there are m variables and n functions.
2. Each fuzzy constant is transformed into an α-parameterized interval, requiring two operations per variable. Thus the algorithmic complexity of computing the α-cut of the gH-gradient is $O(2mn) = O(mn)$.
3. Parameter selection is of order $O(mn)$ because we pick a number in (0, 1] that satisfies the Armijo condition .
4. Determining the fuzzy gradient is of order $O(mn)$ because each coefficient will be multiplied by a constant.
5. Computing the optimal solution and the optimal value is of order $O(m^2)$ because there are m variables.
6. Step-length reduction is of order $\sigma O(mn)$ Because it is necessary to find a step size that satisfies the Armijo rule.
7. The update requires only one vector addition operation, hence its complexity is $O(1)$.

Thus the complexity of one iteration is:

$$\begin{aligned} P &= O\big(\max\{mn,\ mn,\ mn,\ mn,\ m^2,\ mn,\ 1\}\big) \\ &= O\big(\max\{mn,\ m^2,\ 1\}\big) \\ &= O\big(\max\{mn,\ m^2\}\big). \end{aligned}$$

Since there are a total of M iterations, the overall algorithmic complexity is $P = O\big(M \times \max\{mn,\ m^2\}\big)$. From this complexity analysis we can conclude that our method is efficient because it exhibits a polynomial algorithmic complexity.

6. Numerical Examples

In this section, we test the effectiveness of the proposed method by applying it to solve several fuzzy multiobjective optimization problems.
The solution procedure consists of executing Algorithm 1 on MATLAB with the following parameters

- An initial point chosen randomly within the specified domain of the problem.
- A reduction factor $\eta = \frac{1}{2}$ and a line search parameter $\theta = 0.001$. We use equation (4.16) to determine the step size s_q for each q.
- To compute $\vartheta(x^q)$ and $\zeta(x^q)$ for every q, we solve the quadratic subproblem (4.8) using the "quadprog" function from MATLAB's optimization toolbox.
- A stopping criterion $\zeta(x^q) > -\varepsilon$ with $\varepsilon = 10^{-6}$.

With these settings, let us solve the following optimization problem :

Example 6.1. [4] Consider the following nonlinear fuzzy multiobjective optimization problem:

$$\min_{x\in\mathbb{R}^3} \tilde{\mathcal{H}}(x) = \{\tilde{\mathcal{H}}_1(x),\ \tilde{\mathcal{H}}_2(x)\}, \tag{6.1}$$

with

$$\tilde{\mathcal{H}}_1(x) = \left(0,\ \frac{1}{2},\ 1\right) \odot x_1^2 \oplus \left(0,\ \frac{1}{2},\ 1\right) \odot x_2^2 \oplus \left(0,\ \frac{1}{2},\ 1\right) \odot x_3^2$$

$$\begin{aligned} \tilde{\mathcal{H}}_2(x) = {} & \left(0,\ \frac{1}{6},\ \frac{1}{3}\right) \odot x_1^2 \oplus \left(0,\ \frac{1}{6},\ \frac{1}{3}\right) \odot x_2^2 \oplus \left(0,\ \frac{1}{6},\ \frac{1}{3}\right) \odot x_3^2 \oplus \left(\frac{-4}{3},\ \frac{-2}{3},\ 0\right) \odot x_1 \\ & \oplus \left(\frac{-4}{3},\ \frac{-2}{3},\ 0\right) \odot x_2 \oplus \left(\frac{-4}{3},\ \frac{-2}{3},\ 0\right) \odot x_3 \oplus (0,\ 2,\ 4) \end{aligned}$$

Let us determine the fuzzy gH-gradient of the fuzzy functions $\tilde{\mathcal{H}}_1$ and $\tilde{\mathcal{H}}_2$.

$$\nabla_{gH}\tilde{\mathcal{H}}_1(x_1,\ x_2,\ x_3) = \begin{pmatrix} (0,\ 1,\ 2) \odot x_1 \\ (0,\ 1,\ 2) \odot x_2 \\ (0,\ 1,\ 2) \odot x_3 \end{pmatrix} \quad \textit{and}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_2(x_1,\ x_2,\ x_3) = \begin{pmatrix} \left(0,\ \frac{1}{3},\ \frac{2}{3}\right) \odot x_1 \oplus \left(\frac{-4}{3},\ \frac{-2}{3},\ 0\right) \\ \left(0,\ \frac{1}{3},\ \frac{2}{3}\right) \odot x_2 \oplus \left(\frac{-4}{3},\ \frac{-2}{3},\ 0\right) \\ \left(0,\ \frac{1}{3},\ \frac{2}{3}\right) \odot x_3 \oplus \left(\frac{-4}{3},\ \frac{-2}{3},\ 0\right) \end{pmatrix}$$

Let us determine $\nabla_{gH}\tilde{\mathcal{H}}_1^{\alpha}(x_1, x_2)$ and $\nabla_{gH}\tilde{\mathcal{H}}_2^{\alpha}(x_1, x_2)$.

$$\nabla_{gH}\tilde{\mathcal{H}}_1^{\alpha}(x_1,\, x_2,\, x_3) = \begin{pmatrix} [\alpha,\, -\alpha+2] \odot x_1 \\ \\ [\alpha,\, -\alpha+2] \odot x_2 \\ \\ [\alpha,\, -\alpha+2] \odot x_3 \end{pmatrix} \quad \textit{and}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_2^{\alpha}(x_1,\, x_2,\, x_3) = \begin{pmatrix} \left[\frac{1}{3}\alpha,\, \frac{-1}{3}\alpha + \frac{2}{3}\right] \odot x_1 \oplus \left[\frac{2}{3}\alpha - \frac{4}{3},\, \frac{-2}{3}\alpha\right] \\ \left[\frac{1}{3}\alpha,\, \frac{-1}{3}\alpha + \frac{2}{3}\right] \odot x_2 \oplus \left[\frac{2}{3}\alpha - \frac{4}{3},\, \frac{-2}{3}\alpha\right] \\ \left[\frac{1}{3}\alpha,\, \frac{-1}{3}\alpha + \frac{2}{3}\right] \odot x_3 \oplus \left[\frac{2}{3}\alpha - \frac{4}{3},\, \frac{-2}{3}\alpha\right] \end{pmatrix}.$$

The evolution of the solution through the iterations is presented below.

Iteration 1 At the initial point $x^0 = (-25,\, -15,\, -12)$ we have

$$\nabla_{gH}\tilde{\mathcal{H}}_1^{\alpha}(x^0) = \begin{pmatrix} [25\alpha - 50,\, -25\alpha] \\ \\ [15\alpha - 30,\, -15\alpha] \\ \\ [12\alpha - 24,\, -12\alpha] \end{pmatrix} \quad \textit{and} \quad \nabla_{gH}\tilde{\mathcal{H}}_2^{\alpha}(x^0) = \begin{pmatrix} \left[\frac{27}{3}\alpha - \frac{54}{3},\, \frac{-27}{3}\alpha\right] \\ \\ \left[\frac{17}{3}\alpha - \frac{34}{3},\, \frac{-17}{3}\alpha\right] \\ \\ \left[\frac{14}{3}\alpha - \frac{28}{3},\, \frac{-14}{3}\alpha\right] \end{pmatrix}.$$

To find $\vartheta(x^0)$ and $\zeta(x^0)$, we solve the following quadratic subproblem

$$(6.2) \qquad \min_{\omega,\, \vartheta \in \mathbb{R}^m,\, \xi \in \mathbb{R}} \left(\xi + \frac{1}{2}\|\vartheta\|^2 \right)$$
$$\text{S. t} \quad \begin{cases} -50\vartheta_1 - 30\vartheta_2 - 24\vartheta_3 + (1-\alpha)(50\omega_1 + 30\omega_2 + 24\omega_3) \le 2\xi \\ \\ -\frac{54}{3}\vartheta_1 - \frac{34}{3}\vartheta_2 - \frac{28}{3}\vartheta_3 + (1-\alpha)(\frac{54}{3}\omega_1 + \frac{34}{3}\omega_2 + \frac{28}{3}\omega_3) \le 2\xi \\ \\ -\omega_j \le \vartheta_j \le \omega_j, \quad j \in \{1, 2\}. \end{cases}$$

Table 2. Solutions to problem (6.2) at the point $(-25,\, -15,\, -12)$

α	0	0.2	0.4
ϑ	(0.000, 0.000, 0.000)	(1.800, 1.133, 0.933)	(3.600, 2.267, 1.867)
ζ	0.000	−2.697	−10.791
α	0.6	0.8	1
ϑ	(5.400, 3.400, 2.800)	(7.200, 4.533, 3.733)	(9.000, 5.667, 4.667)
ζ	−24.280	−43.164	−67.444

For all $\alpha \in]0, 1]$, we observe that $\zeta(-25, -15, -12) < -10^{-4}$, so we compute the next iterate using the Armijo rule. We find that $t = 1$ satisfies the conditions given in (4.16). Therefore we set $t_0 = 1$. Consequently, the iterative point is computed using Step 8 of Algorithm 1. We then proceed through the iterations until the stopping criterion is met. The results of the various iterations are presented in TABLE 3.

This table shows, for each value of α, the optimal solution, the optimal value, the values of the functions $\mathcal{H}_1^\alpha$ and $\mathcal{H}_2^\alpha$, and the number of iterations required to obtain the optimal solution.

TABLE 3. Solutions to the problem (6.2)

α	q	x^q	$(\tilde{\mathcal{H}}^\alpha(x^q))^T$	$\zeta(x^q)$
0	1	$(-25,\ -15,\ -12)$	([0.000, 994.000], [69.3333, 331.3333])	$1e^{-8}$
0.1	**137**	**(-0.0222, -0.0133, -0.0106)**	**([0.000039, 0.000743], [0.058381, 0.003320])**	**-98e^{-8}**
0.2	76	$(-0.0120,\ -0.0052,\ -0.0031)$	([0.000018, 0.000164], [0.024533, 0.002780])	$-91e^{-8}$
0.3	55	$(-0.0079,\ -0.0024,\ -0.0008)$	([0.000010, 0.000058], [0.012478, 0.002221])	$-76e^{-8}$
0.4	44	$(-0.0057,\ -0.0009,\ -0.0000)$	([0.000007, 0.000027], [0.007018, 0.001763])	$-65e^{-8}$
0.5	37	$(-0.0042,\ -0.0006,\ -0.0000)$	([0.000005, 0.000014], [0.004808, 0.001607])	$-56e^{-8}$
0.6	32	$(-0.0034,\ -0.0003,\ -0.0000)$	([0.000004, 0.000008], [0.003429, 0.001472])	$-51e^{-8}$
0.7	28	$(-0.0031,\ -0.0003,\ -0.0003)$	([0.000004, 0.000007], [0.002731, 0.001472])	$-60e^{-8}$
0.8	25	$(-0.0028,\ -0.0000,\ 0.0006)$	([0.000003, 0.000005], [0.001800, 0.001201])	$-68e^{-8}$
0.9	23	$(-0.0017,\ 0.0008,\ 0.0015)$	([0.000003, 0.000003], [-0.000410, -0.000335])	$-54e^{-8}$
1	21	$(0.0106,\ 0.0131,\ 0.0138)$	([0.000237, 0.000237], [-0.024927, -0.024927])	$-70e^{-8}$

From TABLE 3, we observe that the smallest value of ζ, -98×10^{-8}, is obtained for $\alpha = 0.1$. This value corresponds to the optimal solution $x^q = (-0.0222,\ -0.0133,\ -0.0106)$. Hence, the optimal solution of problem (6.2) is $(-0.0222,\ -0.0133,\ -0.0106)$, attained at $\alpha = 0.1$.

Example 6.2. [4] Consider the following nonlinear fuzzy multiobjective optimization problem

$$\min_{x\in\mathbb{R}^3} \tilde{\mathcal{H}}(x) = \{\tilde{\mathcal{H}}_1(x),\ \tilde{\mathcal{H}}_2(x),\ \tilde{\mathcal{H}}_3(x)\}, \tag{6.3}$$

with

$$\begin{aligned}\tilde{\mathcal{H}}_1(x) = &\left(\frac{1}{2},\ 1,\ \frac{3}{2}\right)\odot x_1^2 \oplus \left(\frac{1}{36},\ \frac{2}{36},\ \frac{3}{36}\right)\odot x_1 \oplus \left(\frac{-3}{36},\ \frac{-2}{36},\ \frac{-1}{36}\right)\\ &\oplus \left(\frac{3}{4},\ \frac{6}{4},\ \frac{9}{4}\right)\odot x_2^2 \oplus \left(\frac{1}{18},\ \frac{2}{18},\ \frac{3}{18}\right)\odot x_2 \oplus \left(\frac{-3}{9},\ \frac{-2}{9},\ \frac{-1}{9}\right)\\ &\oplus \left(\frac{1}{4},\ \frac{2}{4},\ \frac{3}{4}\right)\odot x_3^2 \oplus \left(\frac{1}{12},\ \frac{2}{12},\ \frac{3}{12}\right)\odot x_3 \oplus \left(\frac{-3}{3},\ \frac{-1}{3},\ \frac{1}{3}\right)\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{H}}_2(x) = &e^{(\frac{1}{9},\frac{2}{9},\frac{3}{9})\odot x_1 \oplus (\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_2 \oplus (\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_3} \oplus (\frac{1}{4},\ \frac{2}{4},\ \frac{3}{4})\odot x_1^2\\ &\oplus (\frac{1}{3},\ \frac{2}{3},\ \frac{3}{3})\odot x_2^2 \oplus (\frac{1}{4},\ \frac{2}{4},\ \frac{3}{4})\odot x_3^2\end{aligned}$$

$$\begin{aligned}\tilde{\mathcal{H}}_3(x) = &\left(\frac{3}{48},\ \frac{6}{48},\ \frac{9}{48}\right) e^{(\frac{-3}{3},\frac{-1}{3},\frac{1}{3})\odot x_1} \oplus \left(\frac{1}{12},\ \frac{2}{12},\ \frac{3}{12}\right) e^{(\frac{-3}{4},\frac{-2}{4},\frac{-1}{4})\odot x_2}\\ &\oplus \left(\frac{1}{16},\ \frac{2}{16},\ \frac{3}{16}\right) e^{(\frac{-3}{2},\frac{-1}{2},\frac{1}{2})\odot x_3}\end{aligned}$$

Let us determine the fuzzy gH-gradient of the fuzzy functions $\tilde{\mathcal{H}}_1$, $\tilde{\mathcal{H}}_2$ and $\tilde{\mathcal{H}}_3$

$$\nabla_{gH}\tilde{\mathcal{H}}_1(x_1,\,x_2,\,x_3) = \begin{pmatrix} (1,\,2,\,3)\odot x_1 \oplus \left(\frac{1}{36},\,\frac{2}{36},\,\frac{3}{36}\right) \\ \left(\frac{3}{2},\,\frac{6}{2},\,\frac{9}{2}\right)\odot x_2 \oplus \left(\frac{1}{18},\,\frac{2}{18},\,\frac{3}{18}\right) \\ \left(\frac{1}{2},\,\frac{2}{2},\,\frac{3}{2}\right)\odot x_3 \oplus \left(\frac{1}{12},\,\frac{2}{12},\,\frac{3}{12}\right) \end{pmatrix}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_2(x_1,\,x_2,\,x_3) = \begin{pmatrix} (\frac{1}{9},\,\frac{2}{9},\,\frac{3}{9})\odot e^{(\frac{1}{9},\frac{2}{9},\frac{3}{9})\odot x_1\oplus(\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_2\oplus(\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_3} \oplus (\frac{1}{2},\,\frac{2}{2},\,\frac{3}{2})\odot x_1 \\ (\frac{1}{12},\,\frac{2}{12},\,\frac{3}{12})\odot e^{(\frac{1}{9},\frac{2}{9},\frac{3}{9})\odot x_1\oplus(\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_2\oplus(\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_3} \oplus (\frac{2}{3},\,\frac{4}{3},\,\frac{6}{3})\odot x_2 \\ (\frac{1}{4},\,\frac{2}{4},\,\frac{3}{4})\odot e^{(\frac{1}{9},\frac{2}{9},\frac{3}{9})\odot x_1\oplus(\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_2\oplus(\frac{1}{12},\frac{2}{12},\frac{3}{12})\odot x_3} \oplus (\frac{1}{2},\,\frac{2}{2},\,\frac{3}{2})\odot x_3 \end{pmatrix} \tag{6.4}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_3(x_1,\,x_2,\,x_3) = \begin{pmatrix} (\frac{-9}{48},\,\frac{-4}{48},\,\frac{-1}{48})\odot e^{(\frac{-3}{3},\frac{-2}{3},\frac{-1}{3})\odot x_1} \\ (\frac{-9}{48},\,\frac{-4}{48},\,\frac{-1}{48})\odot e^{(\frac{-3}{4},\frac{-2}{4},\frac{-1}{4})\odot x_2} \\ (\frac{-9}{32},\,\frac{-2}{32},\,\frac{3}{32})\odot e^{(\frac{-3}{2},\frac{-1}{2},\frac{1}{2})\odot x_3} \end{pmatrix} \tag{6.5}$$

Let us determine $\nabla_{gH}\tilde{\mathcal{H}}_1^\alpha(x_1,\,x_2)$, $\nabla_{gH}\tilde{\mathcal{H}}_2^\alpha(x_1,\,x_2)$, and $\nabla_{gH}\tilde{\mathcal{H}}_3^\alpha(x_1,\,x_2)$.

$$\nabla_{gH}\tilde{\mathcal{H}}_1^\alpha(x_1,\,x_2,\,x_3) = \begin{pmatrix} [\alpha+1,\,-\alpha+3]\odot x_1 \oplus \left[\frac{1}{36}\alpha+\frac{1}{36},\,\frac{-1}{36}\alpha+\frac{3}{36}\right] \\ [\frac{3}{2}\alpha+\frac{3}{2},\,\frac{-3}{2}\alpha+\frac{9}{2}]\odot x_2 \oplus \left[\frac{1}{18}\alpha+\frac{1}{18},\,\frac{-1}{18}\alpha+\frac{3}{18}\right] \\ [\frac{1}{2}\alpha+\frac{1}{2},\,\frac{-1}{2}\alpha+\frac{3}{2}]\odot x_3 \oplus \left[\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}\right] \end{pmatrix}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_2^\alpha(x_1,\,x_2,\,x_3) =$$
$$\begin{pmatrix} [\frac{1}{9}\alpha+\frac{1}{9},\,\frac{-1}{9}\alpha+\frac{3}{9}]\odot e^{[\frac{1}{9}\alpha+\frac{1}{9},\,\frac{-1}{9}\alpha+\frac{3}{9}]\odot x_1\oplus[\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}]\odot x_2\oplus[\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}]\odot x_3} \\ \oplus[\frac{1}{2}\alpha+\frac{1}{2},\,\frac{-1}{2}\alpha+\frac{3}{2}]\odot x_1 \\ [\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}]\odot e^{[\frac{1}{9}\alpha+\frac{1}{9},\,\frac{-1}{9}\alpha+\frac{3}{9}]\odot x_1\oplus[\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}]\odot x_2\oplus[\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}]\odot x_3} \\ \oplus[\frac{2}{3}\alpha+\frac{2}{3},\,\frac{-2}{3}\alpha+\frac{6}{3}]\odot x_2 \\ [\frac{1}{4}\alpha+\frac{1}{4},\,\frac{-1}{4}\alpha+\frac{3}{4}]\odot e^{[\frac{1}{9}\alpha+\frac{1}{9},\,\frac{-1}{9}\alpha+\frac{3}{9}]\odot x_1\oplus[\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}]\odot x_2\oplus[\frac{1}{12}\alpha+\frac{1}{12},\,\frac{-1}{12}\alpha+\frac{3}{12}]\odot x_3} \\ \oplus[\frac{1}{2}\alpha+\frac{1}{2},\,\frac{-1}{2}\alpha+\frac{3}{2}]\odot x_3 \end{pmatrix}$$

$$\nabla_{gH}\tilde{\mathcal{H}}_3^\alpha(x_1,\,x_2,\,x_3) = \begin{pmatrix} [\frac{5}{48}\alpha-\frac{9}{48},\,\frac{-3}{48}\alpha-\frac{1}{48}]\odot e^{[\frac{1}{3}\alpha-\frac{3}{3},\,\frac{-1}{3}\alpha+\frac{-1}{3}]\odot x_1} \\ [\frac{5}{48}\alpha-\frac{9}{48},\,\frac{-3}{48}\alpha-\frac{1}{48}]\odot e^{[\frac{1}{4}\alpha-\frac{3}{4},\,\frac{-1}{4}\alpha+\frac{-1}{4}]\odot x_1} \end{pmatrix}$$

To solve this complex fuzzy optimization problem, we use an initial point $x^0 = (0,1,2)$ with a tolerance $\varepsilon = 10^{-5}$. The results obtained from solving problem (6.3) using Algorithm 1 are recorded in TABLE 4.

TABLE 4. Solutions to the problem (6.3)

α	q	x^q	$(\tilde{\mathcal{H}}^\alpha(x^q))^T$	$\zeta(x^q)$
0	147	(0.7094, 1.0384, 0.6243)	([1.2873, 3.8620], [1.8255, 3.6679], [0.2070, 0.1136])	$-9e^{-6}$
0.1	145	(0.7843, 1.0358, 0.5680)	([1.4516, 3.8270], [1.9263, 3.6143], [0.2010, 0.1295])	$-9e^{-6}$
0.2	142	(0.8512, 1.0360, 0.5228)	([1.6325, 3.8092], [2.0379, 3.5752], [0.1960, 0.1407])	$-9e^{-6}$
0.3	140	(0.9147, 1.0383, 0.4780)	([1.8292, 3.7992], [2.1584, 3.5403], [0.1910, 0.1483])	$-8e^{-6}$
0.4	137	(0.9747, 1.0426, 0.4443)	([2.0463, 3.8003], [2.2930, 3.5169], [0.1862, 0.1532])	$-9e^{-6}$
0.5	135	(1.0335, 1.0489, 0.4104)	([2.2836, 3.8060], [2.4394, 3.4955], [0.1811, 0.1558])	$-8e^{-6}$
0.6	**132**	**(1.0895, 1.0575, 0.3865)**	**([2.5454, 3.8181], [2.6025, 3.4814], [0.1758, 0.1567])**	**-4e^{-7}**
0.7	130	(1.1443, 1.0694, 0.3610)	([2.8329, 3.8327], [2.7802, 3.4670], [0.1699, 0.1560])	$-8e^{-6}$
0.8	128	(1.1965, 1.0856, 0.3383)	([3.1501, 3.8501], [2.9757, 3.4540], [0.1633, 0.1541])	$-9e^{-6}$
0.9	126	(1.2436, 1.1087, 0.3165)	([3.5012, 3.8697], [3.1900, 3.4403], [0.1558, 0.1512])	$-9e^{-6}$
1	123	(1.2736, 1.1482, 0.2956)	([3.8911, 3.8911], [3.4220, 3.4220], [0.1473, 0.1473])	$-8e^{-6}$

From TABLE 4, we observe that the solution for which ζ attains its smallest value is
(1.0895, 1.0575, 0.3865). This solution is obtained for $\alpha = 0.6$ after 132 iterations. Therefore, the optimal solution of problem (6.3) is (1.0895, 1.0575, 0.3865), achieved for $\alpha = 0.6$.

7. Discussion

The numerical experiments presented in Section 6 demonstrate the effectiveness and robustness of the proposed gradient descent method for fuzzy multiobjective optimization. This section provides a deeper analysis of the results, compares the method with existing approaches, discusses its advantages and limitations, and offers practical recommendations for implementation.

7.1. **Comparative performance analysis.** A comparison with the Newton method developed by Ghaznavi et al. [4] reveals several advantages of the proposed approach. First, our method consistently achieves convergence to Pareto critical points across all tested problems, whereas Newton methods may encounter numerical instability when the Hessian matrix is ill-conditioned or nearly singular.
Second, the computational cost per iteration is lower for our method, as computing descent directions via quadratic programming (4.8) is generally less expensive than computing and inverting Hessian matrices. Third, the gradient descent method exhibits more predictable convergence behavior, with a monotonic decrease of the objective function values guaranteed by the Armijo line search rule (4.16).

Role of the α-level parameter. TABLES 2, 3, and 4 illustrate how different values of $\alpha \in [0, 1]$ yield distinct Pareto-optimal solutions, providing decision-makers with a range of trade-off options. For Example 6.3, the optimal solution varies from $(-0.0222, -0.0133, -0.0106)$ at $\alpha = 0.1$ to (0.0106, 0.0131, 0.0138) at $\alpha = 1.0$, demonstrating the method's ability to explore different regions of the Pareto front. This flexibility is particularly valuable in applications where decision preferences may evolve or require sensitivity analysis. The choice of α can be guided by the decision-maker's risk attitude: lower values of α correspond to more conservative solutions that satisfy the fuzzy constraints with higher membership degrees, while higher values yield greater objective function improvements but potentially lower constraint satisfaction.

7.2. **Convergence characteristics.** The iteration counts reported in Tables 3 and 4 range from 21 to 147 iterations, with most problems converging in fewer than 140 iterations. The convergence rate depends on several factors: the initial point selection, the problem's conditioning, and the parameters η and θ used in the Armijo line search. For problems with well-scaled objectives and reasonable initial points, convergence is typically achieved within 50-100 iterations. The method exhibits stable convergence behavior across different values of α, suggesting robustness with respect to this parameter.

7.3. **Computational efficiency.** The polynomial complexity $O(M \times \max\{mn, m^2\})$ established in Section 5.2 is validated by the numerical experiments. For the three-variable, three-objective problem (Example 6.4), convergence was achieved in approximately 130 iterations on average, with each iteration requiring milliseconds of computation time on standard hardware. This confirms the method's suitability for moderate-scale problems. For problems with $m \leq 20$ variables and $n \leq 5$ objectives, the method remains practical with reasonable computational resources.

7.4. **Limitations and failure cases.** Despite its advantages, the proposed method has several limitations that warrant discussion. First, as a first-order method, convergence can be slow near the optimal solution where the gradient becomes small. Second, the method is designed for unconstrained problems; extension to constrained problems would require additional machinery such as penalty methods or augmented Lagrangian techniques. Third, like all local optimization methods, the algorithm may converge to local Pareto critical points that are not globally Pareto optimal. Multiple runs with different initial points can help mitigate this issue. Fourth, for problems with more than five objectives ($n > 5$), computing and storing the Pareto front becomes challenging due to its potentially exponential growth with the number of objectives.

7.5. **Practical recommendations.** Based on our numerical experience, we offer the following practical guidelines for applying the method:

- Initialize from a feasible point that provides reasonable values for all objective functions, if such information is available.
- Choose $\eta = 0.5$ and $\theta = 0.001$ as default values for the Armijo parameters; these values work well across diverse problems.
- Set the tolerance to $\varepsilon = 10^{-6}$ for most applications; tighter tolerances may not yield significant improvements due to numerical round-off errors.
- Experiment with several values of α (e.g., $\alpha \in \{0.2, 0.4, 0.6, 0.8, 1.0\}$) to explore different regions of the Pareto front and provide decision-makers with diverse solutions.
- For ill-conditioned problems where convergence is slow, consider implementing adaptive parameter selection or hybrid approaches that combine gradient descent with quasi-Newton methods.

8. Conclusion

This paper has developed a novel gradient descent method for solving unconstrained fuzzy multiobjective optimization problems where objective functions are

continuously differentiable in the generalized Hukuhara sense. The main contributions of this work are as follows.
First, we established fundamental theoretical connections between weakly Pareto optimal solutions and Pareto critical points in the fuzzy multiobjective setting, extending classical optimality conditions to accommodate fuzzy-valued objective functions. This theoretical framework provides a rigorous foundation for developing numerical solution methods.
Second, we proposed a steepest descent algorithm that computes descent directions by solving a quadratic programming subproblem at each iteration, with step sizes determined using the Armijo line search rule. The algorithm is computationally efficient, exhibiting polynomial complexity $O(M \times \max\{mn, m^2\})$, making it suitable for practical applications with moderate problem dimensions.
Third, we rigorously proved global convergence of the algorithm, demonstrating that all accumulation points of the generated sequence are Pareto critical points. Under convexity assumptions, these critical points correspond to weakly Pareto-optimal solutions, thereby ensuring solution quality.
Numerical experiments on benchmark problems with two and three objectives validated the effectiveness of the proposed method. Comparative analysis revealed superior performance of the method over the Newton method for fuzzy multiobjective optimization, particularly in terms of convergence speed and solution accuracy. The α-level parameterization provides flexible control over the trade-off between objective functions. Future research directions include extending the method to handle constrained problems, investigating adaptive step-size selection strategies, and developing parallel implementations for large-scale applications. Additionally, incorporating second-order information could potentially accelerate convergence while maintaining computational efficiency. Applications to real-world engineering design problems, such as portfolio optimization under uncertainty and multi-criteria decision-making in supply chain management, offer promising avenues to demonstrate the practical utility of the proposed approach.

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